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Optimal Control of a Class of Nonlinear Systems: Mixed Integer Programming Based Approach

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Abstract: A mixed integer programming based approach for optimal control of a class of nonlinear systems with typical non-smooth and even discontinuous components is presented. Firstly, a model denoted as General Linear System Model (GLSM), which can be used to describe accurately a wide range of nonlinear systems with typical nonlinear components, is established. Secondly, based on GLSM, mixed integer predictive control (MIPC) approach for optimal regulation and tracking problems is studied. At the end of the paper, numerical simulations are presented. Simulation results show that approach developed in this paper is suitable and effective.

Key Words: Nonlinear System, Optimal Control, Mixed Integer Predictive Control (MIPC), Typical Nonlinear Components.

0 Introduction

Strictly speaking, in the engineering practice, the plants, which the control theories deal with, are all nonlinear, with different degrees of non-linearity. When the plants operate in a very limited range, we can approximate the plants with simple linear models with satisfied accuracy by using the well-known Tyler series expansion approach. But this approach cannot be applied to such a class of nonlinear systems that the systems contain typical nonlinear components which are usually non-smooth, and even are discontinuous, for example, relay, dead zone, magnetic loop. And the series expansion approach cannot be applied to piecewise linear systems either. The also well-known description function method can indeed be employed to treat such a class of nonlinear systems, but this approach can only analysis the self-excited oscillation of such nonlinear system. It is difficult to analysis other response performance of such nonlinear system and is impossible to design controller for nonlinear systems using the description function method. Recently developed feedback linearization method [1] is an effective way for design of controllers for nonlinear systems. But this method also has its limitation: it can only be applied to smooth nonlinear system, it cannot be used to design controller for nonlinear systems with typical nonlinear components.

In this paper, a mixed integer programming based approach for optimal control of a class of nonlinear systems with typical non-smooth and even discontinuous components is presented. Firstly, a model denoted as General Linear System Model (GLSM), which can be used to describe accurately a wide range of nonlinear systems with typical nonlinear components, is established Secondly, based on GLSM, mixed integer predictive control (MIPC) approach for optimal regulation and tracking problems is studied. A mixed integer quadratic programming (MIQP) problem is solved at every time instant when MIPC is implemented. At the end of the paper, numerical example by using the already developed MIQP solver, is presented.

1 General Linear System Model

A model denoted as General Linear System Model (GLSM) is established in this section. In order to simplify the presentation, consider the typical dead zone nonlinear component shown in Fig. 1 when developing the general linear system model. At the end of this section, we will generalize the model for nonlinear systems with other typical nonlinear components.

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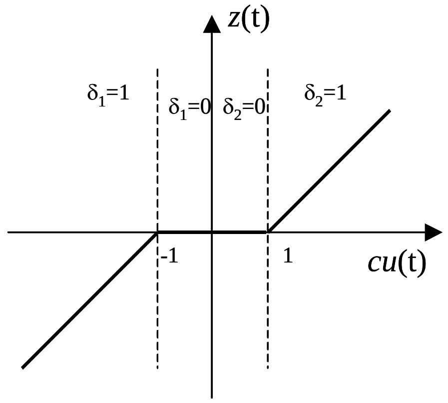

Fig. 1 **dead zone nonlinear component**

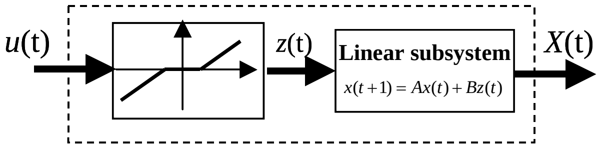

Fig.2a. Cascade connection of linear subsystem with nonlinear components

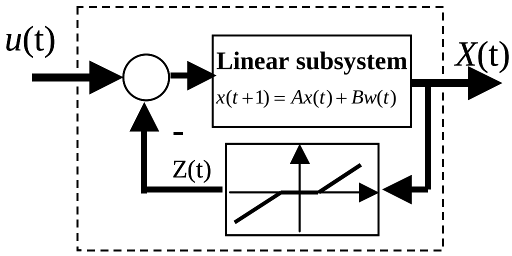

Fig.2b. Feedback connection of linear subsystem with nonlinear components

In Fig.1, assume $u(t) \in R^{m\times 1}$, and $c \in R^{1\times m}$ is constant vector. Introducing auxiliary logical variables $\delta_1(t)$ and $\delta_2(t)$, and their definitions are as in propositional logic (1a) ~ (1d). According to techniques that propositional logic can be equivalently transformed into mixed integer linear inequality [2-5], (1a) ~ (1d) can be rewritten as ineq.(2a) ~ ineq.(2d) respectively.

$$[cu(t) > 1] \Rightarrow [\delta_2(t) = 1] \qquad (1a)$$
$$[cu(t) < -1] \Rightarrow [\delta_1(t) = 1] \qquad (1b)$$
$$[cu(t) < 1] \Rightarrow [\delta_2(t) = 0] \qquad (1c)$$
$$[cu(t) > -1] \Rightarrow [\delta_1(t) = 0] \qquad (1d)$$

$$-cu(t) + 1 \ge \varepsilon + (-M + 1 - \varepsilon)\delta_2(t) \qquad (2a)$$
$$cu(t) + 1 \ge \varepsilon + (m + 1 - \varepsilon)\delta_1(t) \qquad (2b)$$
$$cu(t) - 1 \ge \varepsilon + (m - 1 - \varepsilon)(1 - \delta_2(t)) \qquad (2c)$$
$$-cu(t) - 1 \ge \varepsilon + (-M - 1 - \varepsilon)(1 - \delta_1(t)) \qquad (2d)$$

Where $M \ge \max_{u(t)\in\Theta}\{cu(t)\}$, $m \le \min_{u(t)\in\Theta}\{cu(t)\}$, and values of M, m usually can be estimated reasonably for any specified problem. ε is a very small positive number, for example, 1e-6. According to definitions of $\delta_1(t)$ and $\delta_2(t)$, and with the features of dead zone nonlinear component in the mind, it is easy to obtain (3a) ~ (3b) and (4a) ~(4d). Where $z(t)$ is the output of nonlinear component.

$$[\delta_1(t) = 0] \Rightarrow [z(t) \ge 0] \qquad (3a)$$
$$[\delta_2(t) = 0] \Rightarrow [z(t) \le 0] \qquad (3b)$$

$$z(t) \ge cu(t) - 1 \qquad (4a)$$
$$z(t) \le cu(t) - 1 + (M - m + 1)(1 - \delta_2(t)) \qquad (4b)$$
$$z(t) \le cu(t) + 1 \qquad (4c)$$
$$z(t) \ge cu(t) + 1 + (m - M - 1)(1 - \delta_1(t)) \qquad (4d)$$

It is obvious from (3a) and (3b) that $z(t) = 0$ when $\delta_1(t) = \delta_2(t) = 0$. It can easily be seen from (4a) and (4b) that $z(t) = cu(t) - 1$ when $\delta_2(t) = 1$, and that (4b) is of triviality when $\delta_2(t) = 0$. Again, (3a) and (3b) can be equivalently rewritten as ineq (5a) and (5b) respectively.

$$z \ge \varepsilon + (m - \varepsilon)\delta_1 \qquad (5a)$$
$$z \le \varepsilon + (M - \varepsilon)\delta_2 \qquad (5b)$$

$$V_2\delta(t) + V_3 z(t) \le V_1 u(t) + V_5 \qquad (6)$$

Rearrange the mixed integer linear inequalities (2a) ~ (2d), (4a) ~ (4d), (5a) and (5b), we obtain the vector inequality (6). Where $\delta(t) = [\delta_1(t), \delta_2(t)]'$. V_2, V_3, V_1 *and* V_5 are as follows:

$$V_2 = \left[\begin{array}{c|c} 0 & -M+1-\varepsilon \\ m+1-\varepsilon & 0 \\ 0 & -m+1+\varepsilon \\ M+1+\varepsilon & 0 \\ m-\varepsilon & 0 \\ 0 & -M+\varepsilon \\ 0 & 0 \\ 0 & M-m+1 \\ 0 & 0 \\ -m+M+1 & 0 \end{array}\right] \quad V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad V_1 = \begin{bmatrix} -c \\ c \\ c \\ -c \\ 0 \\ 0 \\ -c \\ c \\ c \\ -c \end{bmatrix} \quad V_5 = \begin{bmatrix} 1-\varepsilon \\ 1-\varepsilon \\ -m \\ M \\ -\varepsilon \\ \varepsilon \\ 1 \\ M-m \\ 1 \\ -m+M \end{bmatrix}$$

In fact, vector inequality (6) established the relationship between the output, the input and the auxiliary logical variables of the nonlinear component. Suppose that a nonlinear plant to be controlled can be modeled as in Fig 2a, the relationship between the system's State $x(t)$ and input $u(t)$ can be established as in (7a) and (7b). We named (7a) and (7b) as the General Linear System Model, GLSM.

$$\begin{cases} x(t+1) = Ax(t) + Bz(t) & (7a) \\ V_2\delta(t) + V_3 z(t) \le V_1 u(t) + V_5 & (7b) \end{cases}$$

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) - Bz(t) & (8a) \\ V_2\delta(t) + V_3 z(t) \le V_1 x(t) + V_5 & (8b) \end{cases}$$

In the GLSM, $z(t)$ act as the auxiliary continuous variable. At first glimpse, it seems that the GLSM is linear. But in fact, because there exist the restrictions of the values (0 or 1) for logical variables $\delta_1(t)$ and $\delta_2(t)$, GLSM is nonlinear. In addition to modeling systems containing the dead zone nonlinear component, GLSM can also describe accurately systems with other typical nonlinear components shown in Fig 3.

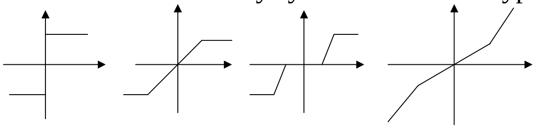

Fig. 3 several non-smooth and even discontinuous components

$$\begin{cases} x(t+1) = Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) & (9a) \\ y(t) = Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) & (9b) \\ E_2\delta(t) + E_3z(t) \le E_1u(t) + E_4x(t) + E_5 & (9c) \end{cases}$$

If a plant to be controlled can be modeled as in Fig 2b, the relationship between the system's State $x(t)$ and input $u(t)$ can be established as in (8a) and (8b).

To generalize (7a), (7b) and (8a), (8b), we obtained the GLSM as in (9a) ~ (9c). It can be pointed out that the GLSM can easily deal with the system's states and/or inputs constraints which usually had to be satisfied because of, for example, the physical restricts or for the operation safety.

2 Optimal Control of Nonlinear Systems

Based on GLSM, Mixed Integer Predictive Control, MIPC, approach for optimal regulation and tracking problems of nonlinear systems is studied in this section. A Mixed Integer Quadratic Programming, MIQP, problem is solved at every time interval when MIPC is implemented. Suppose that t is current time, and that $x(t)$ is the current system state. Denote $u_t^{k-1} = \{u_t(0), u_t(1), ..., u_t(k-1)\}$, to be the predicted optimal control sequence and $x(k \mid t) = x(t+k, x(t), u_t^{k-1})$ to be the predicted system state at $t+k$. $\delta(k \mid t)$, $z(k \mid t)$, $x(k \mid t)$, $y(k \mid t)$ are similarly defined. At time t, establishing following optimization problem (12) and (13):

$$\min_{\{u_t^{T-1}\}} J(u_t^{T-1}, x(t)) = \min_{\{u_t^{T-1}\}} \{\sum_{k=0}^{T-1} \|u_t(k) - u_e\|_{Q1}^2 + \|\delta(k \mid t) - \delta_e\|_{Q2}^2 + \|z(k \mid t) - z_e\|_{Q3}^2 + \|x(k \mid t) - x_e\|_{Q4}^2 + \|y(k \mid t) - y_e\|_{Q5}^2\} \quad (12)$$

$$\text{subject to:} \begin{cases} x(k+1 \mid t) = Ax(k \mid t) + B_1u_t(k) + B_2\delta(k \mid t) + B_3z(k \mid t) & (13a) \\ y(k \mid t) = Cx(k \mid t) + D_1u_t(k) + D_2\delta(k \mid t) + D_3z(k \mid t) & (13b) \\ E_2\delta(k \mid t) + E_3z(k \mid t) \le E_1u_t(k) + E_4x(k \mid t) + E_5 & (13c) \\ x(T \mid t) = x_e, \qquad \text{k = 0,1,....., T - 1} & (13d) \end{cases}$$

Where $\|x\|_{Qi}^2 = x'Q_ix$. $Qi = Qi' \ge 0$ is weighted matrix. And (x_e, u_e, δ_e, z_e) satisfying (14) is the equilibrium point of GLSM.

$$\begin{cases} x_e = Ax_e + B_1u_e + B_2\delta_e + B_3z_e \\ E_2\delta_e + E_3z_e \le E_1u_e + E_4x_e + E_5 \end{cases} \quad (14)$$

In order to guarantee the stability of the closed loop mixed integer predictive control system, we introduced the equality constraint (13d) in the optimization problem. From (13a), we can obtained (15)

$$x(k \mid t) = A^k x(t) + \sum_{i=0}^{k-1} A^i [B_1u(k-1-i) + B_2\delta(k-1-i \mid t) + B_3z(k-1-i \mid t)] \quad (15)$$

By introducing the follow denotations and, after rearrangement of the (12), (13) and (15), we obtained (16), which is the standard Mixed Integer Quadratic Programming problem, MIQP.

$$\Omega = \begin{bmatrix} u_t(0) \\ u_t(1) \\ \vdots \\ u_t(T-1) \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta(0 \mid t) \\ \delta(1 \mid t) \\ \vdots \\ \delta(T-1 \mid t) \end{bmatrix}, \quad \psi = \begin{bmatrix} z(0 \mid t) \\ z(1 \mid t) \\ \vdots \\ z(T-1 \mid t) \end{bmatrix}, \quad \gamma = \begin{bmatrix} \Omega \\ \Delta \\ \psi \end{bmatrix},$$

$$\min_{\gamma} \{ \gamma' H\gamma + F\gamma \}$$

$$\text{Subject to:} \begin{cases} A_{ineq}\gamma \le b_{ineq} \\ A_{eq}\gamma = b_{eq} \\ \gamma \in R^{nc} \times \{0,1\}^{nd} \\ \gamma(itype) \in \{0,1\}^{nd} \end{cases} \quad (16)$$

Suppose we obtained the optimal control sequence $u_t^{*\,T-1} = \{u_t^*(0), u_t^*(1), ..., u_t^*(T-1)\}$ at current time t by solving the MIQP problem of (16). According to the philosophy of the receding horizon control [6], only the optimal control $u_t^*(0)$ is actually enforced upon the plant, and other control $\{u_t^*(1), u_t^*(2), ..., u_t^*(T-1)\}$ is discarded. In the next sampling time t+1, repeat the above steps of (12)~(16) when $x(t+1)$ is available.

For optimal regulating problem, the above Mixed Integer Predictive Control can regulate optimally the initial state $x_0(t)$ to system' equilibrium point (x_e, u_e, δ_e, z_e) which is usually the origin.

For optimal tracking problem, the output of the system, $y(t)$, can track optimally reference input $r(t)$. In this case, equilibrium point ($x_{et}, u_{et}, \delta_{et}, z_{et}$) for any specific time t and corresponding $r(t)$, can be obtained as follows:

$$\min_{\{xet,uet,zet,\delta et\}} \{\rho(\|x_{et}\|^2 + \|\delta_{et}\|^2 + \|z_{et}\|^2 + \|u_{et}\|^2) + \|y_{et} - r(t)\|_Q^2\} \quad (17)$$

$$\text{subject to: } \begin{cases} x_{et} = Ax_{et} + B_1u_{et} + B_2\delta_{et} + B_3z_{et} \\ E_2\delta_{et} + E_3z_{et} \le E_1u_{et} + E_4x_{et} + E_5 \\ y_{et} = Cx_{et} + D_1u_{et} + D_2\delta_{et} + D_3z_{et} \end{cases} \quad (18)$$

where ρ and Q are weighted positive numbers. (17) and (18) can also be cast as a standard MIQP problem (16). The MIQP problem can be solved effectively by using the Branch & Bound (B&B) approach [7].

3. Numerical Example

Suppose that the linear subsystem of a nonlinear plant to be $x(t+1) = \begin{bmatrix} 1 & 0.091 \\ 0 & 0.819 \end{bmatrix} x(t) + \begin{bmatrix} 0.005 \\ 0.091 \end{bmatrix} w(t)$, and that the plant can be modeled as in Fig. 2b. c=[3 3], $\rho = 1, Q = 5000$, x0=[-1 1]', T=3;

$r(t) = \sin(\frac{2\pi}{50}t)$, $Q1 = 0.01, Q2 = 0.01 * \mathrm{I}(2,2), Q3 = 0.01, Q4 = 0.1 * I(2,2), Q5 = 1000$, and I(2,2) is identity. Simulation results are shown in Fig. 4a and Fig.4b.

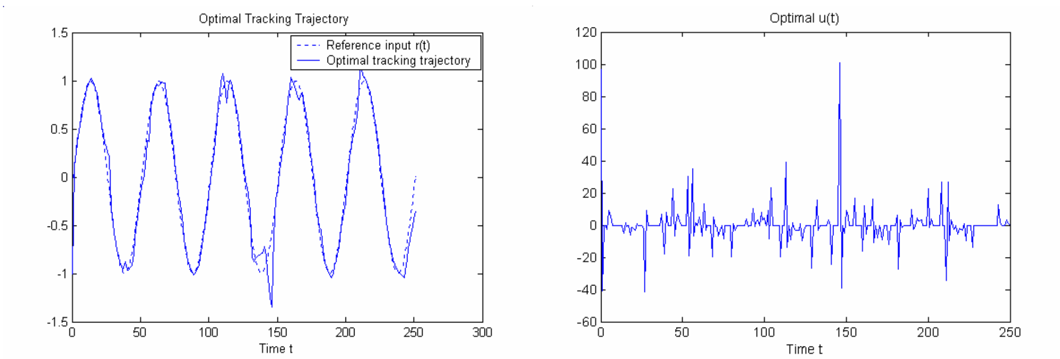

Fig. 4a, Optimal tracking trajectory **Fig. 4b, Optimal control inputs**

4. Conclusions

A mixed integer programming based approach for optimal control of a class of nonlinear systems with typical non-smooth and even discontinuous components is presented. A model denoted as General Linear System Model (GLSM) is established which can describe accurately systems with typical nonlinear components, for example, relay, dead zone, magnetic loop. Numerical simulations show that approach developed in this paper is suitable and effective for optimal control of a class of nonlinear systems.

References

[1] Feng Chunbo, Fei shuming。 Analysis and design of nonlinear control system (2nd Edition). Publishing house of electrical industry。 Beijing: 1998。

[2] Tyler,M.L. & Morrari M. propositional logic in control and monitoring problems. Automatica 35(1999) 562-582。

[3] Bemporad A., & Morrari M. control of systems integrating logic, dynamics, and constraints Automatica 35(1999) 407-427。

[4] William, H.P. Linear and integer programming applied to the propositional calculus. Bull. Institute Math. Appli. 13(1977),18-20。

[5] Cavalier, T.M Modeling and integer programming techniques applied to propositional calculus. Comput. Oper. Res. 17(6), 561-570。

[6] Morari M., Lee J.H. Model predictive control: past, present and future. Computer and Chemical Engineering Vol.23 667-682。

[7] Zhang ju, Li ping and Wan wangliang. Branch & Bound Based Solution of MIQP Problem and its Application, Journal of system simulation Vol.15(4),2003。