

# Substitutions and properties of robustness and non-fragility in controllers

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*Abstract:* In this paper, using algebraic methods, we characterize the parameters of a linear fractional transformation such that the composition of a class of rational function with the linear fractional transformation preserves stability, in the case that the rational function is stable, or stabilizes the original rational function, in the case that the rational function is unstable. As a consequence, we obtain a dual result about the robust stabilization of a plant—represented as a rational function—compensated with a controller when there is a nonlinear disturbance induce by function composition on the parameters of the controller. This implies the non-fragility of the controller and also the plant robust stabilization for the same class of disturbances. Also, for a particular choice of one of the parameters in the linear fractional transformation, the composition of functions preserves the structure of Proportional, Proportional-Derivative and Proportional-Derivative-Integral type of controllers.

*Key-Words:* Linear fractional transformation, composition, robust stabilization, PD/PI/PID controllers

## 1 Introduction

Recently in literature it has appeared a series of articles on the subject of preservation of stability for linear systems in the frequency domain, [1, 3, 4, 5, 8]. In [5] it is presented a method on maps that preserve the stability of stable polynomials, i.e., the map that is obtained by multiplying the vector of coefficients of stable polynomials by a fixed matrix to obtain a vector of stable coefficients. This method does not have a complete characterization of which matrices preserve stability. Other methods, used a substitution of a rational function in a polynomial to guarantee stability and are based on  $H$ -domains and diagrams of Mikhailov [8]. In [1], it is used the substitution  $\alpha(s) = \frac{as+b}{cs+d}$  for the  $s$  variable in a stable rational function and it is proven that for positive real numbers  $a$ ,  $b$ ,  $c$  and  $d$ , such that  $ad - bc \neq 0$ , this substitution preserves stability, but the case is very restrictive. In [3, 4], the results from [1] are extended and generalized, showing that substitutions of the  $s$  variable in a rational stable function by a strictly positive real functions with relative degree zero, preserve stability, and under some additional conditions, powers of functions SPR0, also preserve stability, but only

sufficient conditions are given. In this work using algebraic methods, we give a complete characterization on the parameters of a linear fractional transformation,  $\alpha(s) = \frac{as+b}{cs+d}$ , such that the composition of a class of rational, real, proper, stable or unstable functions,  $H(s) = \frac{N_h(s)}{D_h(s)}$ , with the linear fraction transformation  $\alpha(s)$  is stable, i.e., find the parameters  $a$ ,  $b$ ,  $c$  and  $d$  such that  $\overline{H}(\alpha(s))$  is stable, with  $\alpha(s) = \frac{as+b}{cs+d}$ . These results generalize and extend previous results [1, 2, 3, 4]. In addition, it is possible to answer the open problem proposed in [5] for the case of maps that preserve stability, obtained under the substitution of the variable  $s$  by  $\alpha(s)$  in a stable polynomial. This is done by characterizing all the maps obtained under this substitution, that preserve stability for any stable polynomial  $D_h(s)$  which is mapping to stable polynomial  $(cs + d)^m D_h(\alpha(s))$ . In other words, we characterized the space of parameters  $a$ ,  $b$ ,  $c$  and  $d$  for which the map, preserves stability for any stable polynomial, mapping stable polynomials in stable polynomials. But also we characterized the space of parameters  $a$ ,  $b$ ,  $c$  and  $d$  for which the map, stabilizes unstable polynomials. As a consequence, we obtain a dual result, in the sense that the robust stabiliza-

tion of a plant  $H(s)$  with disturbances induced by the substitution of the variable  $s$  by  $\alpha^{-1}(s)$ , with a controller  $C(s)$ , implies the non-fragility of the controller  $C(s)$  under the same class of disturbances, induced by the substitution of the variable  $s$  by  $\alpha(s)$ , in the controller, and vice versa i.e., the non-fragility of the controller  $C(s)$  under disturbances induced by the substitution of the variable  $s$  by  $\alpha^{-1}(s)$ , implies the robust stabilization of a plant  $H(s)$  with disturbances, induced by the substitution of the variable  $s$  by  $\alpha(s)$  with a controller  $C(s)$ . In the particular case when  $b = 0$ , the substitution of the  $s$  variable by  $\alpha(s)$ , preserves the structure for Proportional-Derivative (PD), Proportional-Integral (PI) and Proportional-Integral-Derivative (PID) controllers. Based on the resulting pseudo-parametrization for these controllers class after the substitution of the variable  $s$  by  $\alpha(s)$ , taking  $b = 0$ , we mention some ideas that could be used later for tuning rules on the derivative part for the PD controllers, and for the proportional and integral parts of the PI controllers. Finally, we give a result about stabilization based on passivity.

## 2 Preliminaries

This section we give the necessary definitions and notation used throughout the paper.

Let  $\mathbf{R}(s)$  be the set of rational functions with real coefficients. Consider a rational function  $H(s) \in \mathbf{R}(s)$

$$H(s) = \frac{N_h(s)}{D_h(s)} = k \frac{s^n + a_{n-1}s^{n-1} + \dots + a_0}{s^m + b_{m-1}s^{m-1} + \dots + b_0}$$

where  $N_h(s)$  and  $D_h(s)$  are coprime, with  $m \geq n$ . Let us factorize  $H(s)$  as  $H(s) = H_r(s)H_c(s)$ , where

$$H_r(s) = k \frac{(s - z_1) \dots (s - z_{n-l_1})}{(s - p_1) \dots (s - p_{m-j_1})}$$

has real poles and zeros,  $l_1 < n$ ,  $j_1 < m$ ; and

$$H_c(s) = \frac{\left[ (s - \rho_1)^2 + \nu_1^2 \right] \dots \left[ (s - \rho_{l_0})^2 + \nu_{l_0}^2 \right]}{\left[ (s - \sigma_1)^2 + \omega_1^2 \right] \dots \left[ (s - \sigma_{j_0})^2 + \omega_{j_0}^2 \right]}$$

has complex poles and zeros,  $l_0 = \frac{l_1}{2}$  and  $j_0 = \frac{j_1}{2}$ .

**Definition 1 ([6, 7])** A real and rational function  $H(s)$  is strictly positive real (SPR) if  $H(s)$  is analytic in  $\text{Re}[s] \geq 0$  and  $\text{Re}[H(j\omega)] > 0$  for all  $\omega \in \mathbb{R}$ , where  $j = \sqrt{-1}$ . Moreover, a real and rational function  $p(s)$  is SPR0 if it is SPR and has zero relative degree.

Let us define the following sets:

$$\text{SPR0}^* = \left\{ p(s) \in \mathbf{R}(s) : p(s) \text{ is SPR0} \right\} \cup \{s\}.$$

$$\Gamma(a, b, c, d) = \left\{ \alpha(s) \in \mathbf{R}(s) : \alpha(s) = \frac{as + b}{cs + d}, \right. \\ \left. ad - bc \neq 0 \text{ and } a, b, c, d > 0 \right\} \cup \{s\}.$$

The following properties can be easily verified for the set  $\Gamma(a, b, c, d)$

1.  $\lim_{(b,c) \rightarrow (0,0)} \frac{as+b}{cs+a} = s$ , where  $a^2 - bc > 0$ ;
2. if  $\alpha(s), \beta(s) \in \Gamma(a, b, c, d)$ , then  $\alpha(\beta(s)), \beta(\alpha(s)) \in \Gamma(a, b, c, d)$ .

From the associative property of function composition, we know that the set  $\Gamma(a, b, c, d)$  is a non commutative monoid under the composition operation. Additionally, it is well known that  $\Gamma(a, b, c, d) \subset \text{SPR0}^*$ , [1].

## 3 On the preservation of stabilization, fragility and passivity in PI, PD and PID controllers

Consider a Single-Input Single-Output (SISO) Linear Time-Invariant (LTI) system with state variable representation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (1)$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^n$ ,  $C \in \mathbb{R}^{1 \times n}$  and  $D \in \mathbb{R}$ ; and transfer function representation

$$H(s) = C(sI - A)^{-1} + D$$

We use a linear fraction transformation,  $\alpha(s) = \frac{as+b}{cs+d}$ , to obtain the SISO LTI system  $H(\alpha(s))$  with state variables representation

$$\begin{aligned} \dot{x} &= (dA - bI)Q_A x + (ad - bc)Q_A Bu \\ y &= CQ_A x + (D + cCQ_A B)u \end{aligned} \quad (2)$$

with  $Q_A = (aI - cA)^{-1}$ .

The problems to study in this section are the following:

1. If system (1) is stable, for which set of parameters  $a, b, c, d$ , system (2) is stable?
2. If system (1) is unstable, for which set of parameters  $a, b, c, d$ , system (2) is stable?

The answers to the previous questions are given in the following two results.

**Lemma 2** Consider a plant  $H(s) = \frac{N_h(s)}{D_h(s)}$  where  $N_h(s)$  and  $D_h(s)$  are polynomials satisfying  $\deg D_h(s) = m \geq \deg N_h(s) = n$ . Let us also define the linear fractional transformation  $\alpha(s) = \frac{as+b}{cs+d}$  where  $a, b, c$  and  $d$  are real numbers such that  $cd \neq 0$  and  $ad - bc \neq 0$ . Let us substitute the  $s$  variable by  $\alpha(s)$  in  $H(s)$ , i.e.,  $H(\alpha(s))$ . Then,  $H_\alpha(s) \equiv H(\alpha(s))$  is stable if and only if the following conditions holds:

1. Either  $p_i d - b > 0$  and  $a - p_i c < 0$ , or  $p_i d - b < 0$  and  $a - p_i c > 0$  for each  $i = 1, \dots, m - j_1$  where  $p_1, \dots, p_{m-j_1}$  are the real poles of  $H(s)$ ;
2. the parameters  $a, b, c$  and  $d$  satisfies

$$\sigma_i^2 - \left( \frac{a}{c} + \frac{b}{d} \right) \sigma_i + \frac{ab}{cd} + \omega_i^2 > 0$$

for  $i = 1, \dots, j_0$ , where  $\sigma_i + j\omega_i$  are the complex poles of  $H(s)$ .

**Remark 3** Notice that if we consider  $x = \frac{a}{c}$  and  $y = \frac{b}{d}$ , the function  $f(x, y) = \sigma_j^2 - (x + y) \sigma_j + xy + \omega_j^2$  has a local minimum at  $x = y = \sigma_j$ , but  $f(x, y)$  does not have a global minimum for  $x, y \in \mathbb{R}$ .

Note that the parameters  $a, b, c$  and  $d$  can be negative, and  $c$  and  $d$  must be different from zero. Moreover, no assumption on the sign of the poles of  $H(s)$  is made.

The case when one or two of the parameters  $a, b, c$  and  $d$  are zero or negative is considered in the following result. It is clear that there exist only two cases that make sense for two parameters equal to zero and none for more than two parameters equal to zero.

**Lemma 4** Consider  $H(s) = \frac{N_h(s)}{D_h(s)}$  as defined in Lemma 2, but stable, (i.e.,  $p_1, \dots, p_{m-j_1} < 0$ ,  $\sigma_1, \dots, \sigma_{j_0} < 0$ ). Then  $H(\alpha(s))$  is a stable plant if  $\alpha(s)$  and  $H(s)$  satisfies at least one of the following conditions:

1.  $a, b, c, d > 0$  and  $ad - bc \neq 0$ , or  $a, b, c, d < 0$  and  $ad - bc \neq 0$ ;
2.  $b, c, d > 0, a = 0$  or  $b, c, d < 0, a = 0$ ; and all the poles in  $H(s)$  must be complex.
3.  $a, c, d > 0, b = 0$  or  $a, c, d < 0, b = 0$ ;
4.  $a, b, d > 0, c = 0$ ;
5.  $a, b, c > 0, d = 0$ , and  $\max \{ \sigma_1, \dots, \sigma_{j_0} \} < \frac{a}{c}$ ;

6.  $a, d > 0, b = c = 0$  or  $a, d < 0, b = c = 0$ ;
7.  $b, c > 0, a = d = 0$  or  $b, c < 0, a = d = 0$ ; and all the poles in  $H(s)$  must be real.
8.  $a, b > 0, d < 0, c = 0$  and  $p_i d - b < 0$  for  $i = 1, \dots, m - j_1$  and  $\max \{ \sigma_1, \dots, \sigma_{j_0} \} < \frac{b}{d}$ ;
9.  $a, b < 0, d < 0, c = 0$ ;
10.  $a < 0, b > 0, d < 0, c = 0$  and  $p_i d - b > 0$  for  $i = 1, \dots, m - j_1$  and  $b - d\sigma_j > 0$  for  $j = 1, \dots, j_0$ ;
11.  $a, b > 0, c < 0, d = 0, a - p_i c > 0$  for  $i = 1, \dots, m - j_1$  and  $a - \sigma_j c > 0$  for  $j = 1, \dots, j_0$ ;
12.  $a, b < 0, c < 0, d = 0$ ;  $a, b, c > 0, d = 0$ .
13.  $a > 0, b < 0, c < 0, d = 0$  and  $a - p_i c < 0$  for  $i = 1, \dots, m - j_1$  and  $a - c\sigma_j > 0$  for  $j = 1, \dots, j_0$ .

**Remark 5** When  $a > 0, b < 0, d < 0, c = 0$  or  $a < 0, b > 0, c < 0, d = 0$  then stability is not guarantee, and stable plants are not mapped into stable plants, unless  $p_1, \dots, p_{m-j_1} > 0$  and  $\sigma_1, \dots, \sigma_{j_0} > 0$ .

**Lemma 6** In the case when  $p_1, \dots, p_{m-j_1} > 0$  and  $\sigma_1, \dots, \sigma_{j_0} > 0$  the plant  $H(\alpha(s))$  is a stable if at least one of the following conditions holds:

1.  $a > 0, b < 0, d < 0, c = 0$  and  $p_i d - b < 0$  for  $i = 1, \dots, m - j_1$  and  $b - d\sigma_j > 0$  for  $j = 1, \dots, j_0$ ;
2.  $a < 0, b > 0, c < 0, d = 0$  and  $a - p_i c < 0$  for  $i = 1, \dots, m - j_1$  and  $a - c\sigma_j > 0$  for  $j = 1, \dots, j_0$ .

Now we are going to present some applications of the former technical results to the duality between robust stabilization and fragility of controllers.

**Proposition 7** Let us consider the proper plant  $H(s) = \frac{N_h(s)}{D_h(s)}$  and the proper controller  $C(s) = \frac{N_c(s)}{D_c(s)}$  such that it stabilizes the plant, where  $N_h(s), N_c(s), D_c(s)$  and  $D_h(s)$  are polynomials with  $\deg D(s) = n \geq \deg N(s)$ . Also consider the linear transformation  $\alpha(s) = \frac{as+b}{cs+d}$  where  $a, b, c, d \in \mathbb{R}$ , and let us substitute the  $s$  variable by  $\alpha^{-1}(s) = \frac{b-ds}{cs-a}$  in  $H(s)$ . Then:

1. the controllers of the form  $C_\alpha(s) \equiv C(\alpha(s))$  stabilizes  $H(s)$ , if  $C(s)$  stabilizes in a robust way the plant  $H_{\alpha^{-1}}(s) \equiv H(\alpha^{-1}(s))$ , where the  $a, b, c, d$  parameters satisfy at least one of

the conditions of Lemma 4, or the conditions of Lemma 2 for the closed loop plant:

$$\bar{P}(s) = \frac{C(s)H_{\alpha^{-1}}(s)}{1 + C(s)H_{\alpha^{-1}}(s)}.$$

- the controller  $C(s)$  stabilizes in a robust way the plants  $H_{\alpha}(s) \equiv H(\alpha(s))$ , if the controllers  $C(\alpha^{-1}(s))$  stabilizes the plant  $H(s)$ , where the  $a, b, c, d$  parameters satisfies at least one of the conditions of Lemma 4, or the conditions of Lemma 2 for the closed loop plant:

$$\hat{P}(s) = \frac{C(\alpha^{-1}(s))H(s)}{1 + C(\alpha^{-1}(s))H(s)}.$$

Now, for the particular case, when the controller is a PD, PI, or PID is studied.

Let us consider a PD controller of the form  $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$ , and a PI controller of the form  $C_{PI}(s) = K_p + \frac{K_I}{s}$ . These controllers can be rewritten as:  $C_{PD}(s) = \frac{(K_p + K_D)s + K_p r}{s+r}$  and  $C_{PI}(s) = \frac{K_p s + K_I}{s}$ . Note that  $C_{PD}(s) \in \Gamma(a, b, c, d)$  if  $K_p, K_D, r > 0$ . We can now attack the problem of the non-fragile stabilization under non linear disturbances induced by the substitution of the  $s$  variable by the linear fractional transformation  $\alpha(s)$ .

We then have the following results:

### Corollary 8

- If the controller  $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$  robustly stabilizes the plants  $H_{\alpha^{-1}}(s)$ , where the parameters  $a, b, c, d$  satisfies at least one of the conditions of Lemma 4 for the closed loop system formed by  $C_{PD}(s)$  and  $H_{\alpha^{-1}}(s)$ , then the controllers

$$C_{PD}(\alpha(s)) = \left( \frac{(K_p + K_D)a + K_p r c}{a + r c} \right) \frac{s + \frac{b+ld}{a+lc}}{s + \frac{b+rd}{a+rc}}$$

stabilizes  $H(s)$  with  $l = \frac{K_p r}{K_p + K_D}$ . If  $\alpha(s) \in \Gamma(a, b, c, d)$ , and  $K_p, K_D, r > 0$ , then  $C_{PD}(\alpha(s)) \in \Gamma(a, b, c, d)$ .

- If the controller  $C_{PI}(s) = K_p + \frac{K_I}{s}$  robustly stabilizes the plants  $H_{\alpha^{-1}}(s)$ , where the parameters  $a, b, c, d$  satisfies at least one of the conditions of Lemma 4 for the closed loop system formed by  $C_{PI}(s)$  and  $H_{\alpha^{-1}}(s)$ , then the controllers

$$C_{PI}(\alpha(s)) = \left( \frac{K_p a + K_I c}{a} \right) \frac{s + \frac{K_p b + K_I d}{K_p a + K_I c}}{s + \frac{b}{a}}$$

stabilizes  $H(s)$ . If  $\alpha(s) \in \Gamma(a, b, c, d)$  and  $K_p, K_I > 0$ , then  $C_{PI}(\alpha(s)) \in \Gamma(a, b, c, d)$ .

Note that the controllers  $C_{PD}(\alpha(s))$  and  $C_{PI}(\alpha(s))$  in Corollary 8 are not PD or PI controllers (unless  $b = 0$ ). Both controllers are lead-lag networks. As  $\Gamma(a, b, c, d) \subset \text{SPR0}^*$ , then  $C_{PD}(\alpha(s))$  and  $C_{PI}(\alpha(s))$  are strictly passive controllers. Obviously, they are also a dual version of this result.

When the substitution is  $\gamma(s) = \frac{as}{cs+d}$  we then have the following interesting result:

### Corollary 9

- If  $C_{PD}(s) = K_p + \frac{K_D s}{s+r}$  robustly stabilizes the family  $H_{\gamma^{-1}}(s)$ , then the PD controllers:

$$\hat{C}_{PD}(s) = K_p + \frac{\hat{K}_D s}{s + q}$$

with  $\hat{K}_D = \frac{aK_D}{a+rc}$  and  $q = \frac{rd}{a+rc}$ , stabilizes  $H(s)$ , for any real  $a, b, c, d$  such that  $a, c, d > 0$  and  $b = 0$ ;

- If  $C_{PI}(s) = K_p + \frac{K_I}{s}$  robustly stabilizes the family  $H_{\gamma^{-1}}(s)$ , then the PI controllers:

$$\hat{C}_{PI}(s) = \hat{K}_p + \frac{\hat{K}_I}{s}$$

with  $\hat{K}_p = K_p + \frac{K_I c}{a}$  and  $\hat{K}_I = \frac{dK_I}{a}$ , stabilizes  $H(s)$ , for any real  $a, b, c, d$  such that  $a, c, d > 0$  and  $b = 0$ ;

- If  $C_{PID}(s) = \left( K_p + \frac{K_I}{s} \right) \left( K_p + \frac{K_D s}{s+r} \right)$  robustly stabilizes the family  $H_{\gamma^{-1}}(s)$ , then the PID controllers:

$$\hat{C}_{PID}(s) = \left( \hat{K}_p + \frac{\hat{K}_I}{s} \right) \left( K_p + \frac{\hat{K}_D s}{s + q} \right)$$

with  $\hat{K}_p = K_p + \frac{K_I c}{a}$ ,  $\hat{K}_I = \frac{dK_I}{a}$  and  $\hat{K}_D = \frac{aK_D}{a+rc}$ , stabilizes  $H(s)$ , for any real  $a, b, c, d$  such that  $a, c, d > 0$  and  $b = 0$ .

Clearly, the substitution  $\gamma(s)$ , preserves the structure of the PD and PI controllers. In the case of PD controllers it is interesting to note that the  $K_p$  constant doesn't change. This can be interpreted in the following way: the predictive part of the PD controller can be modified following the relations:

$$\hat{K}_D = \frac{aK_D}{a + rc}$$

$$q = \frac{rd}{a + rc}$$

They can be seen as a pseudo-parametrization of the derivative part. We can then use this information to

develop in the future tuning rulers for the derivative part of the controller. In the same way, we can see that in the case of PI controllers the gains change following

$$\widehat{K}_p = K_p + \frac{K_I c}{a}$$

$$\widehat{K}_I = \frac{d K_I}{a}$$

By using standard results on passivity, we can to give the following result.

**Corollary 10** Consider the following controllers:

1.  $C_1(s) = C_{PI}(s) = K_p + \frac{K_I}{s}$  where  $K_p, K_I > 0$ .
2.  $C_2(s) = C_{PD}(s) = K_p + \frac{K_D s}{s+r}$  where  $r, K_p, K_D > 0$ .
3.  $C_3(s) = C_{LL}(s) = K_p \frac{1+T_N s}{1+T_D s}$  where  $K_p, T_D, T_N > 0$ .
4.  $C_4(s) = C_{PID_1}(s) = K_p + \frac{K_I}{s} + \frac{K_D s}{s+r}$  where  $r, K_p, K_I, K_D > 0$ .
5.  $C_5(s) = C_{PID_2}(s) = K_p + \frac{K_I}{s} + K_D s$  where  $K_p, K_I, K_D > 0$ .
6.  $C_6(s) = C_{PID_3}(s) = K_p \left( \frac{1+T_i s}{T_i s} \right) \frac{1+T_d s}{1+\eta T_d s}$  where  $K_p > 0, 0 < T_d < T_i$  and  $0 < \eta \leq 1$ .
7.  $C_7(s) = C_{PID_4}(s) = K_p \beta \left( \frac{1+T_i s}{1+\beta T_i s} \right) \frac{1+T_d s}{1+\eta T_d s}$  where  $K_p > 0, 0 < T_d < T_i, 1 \leq \beta$  and  $0 < \eta \leq 1$ .

Now the following assumption is made: Given a fixed plant  $H(s)$ , there exists a subset  $\Omega$  of linear transformations  $\alpha(s) = \frac{as+b}{cs+d}$  where  $a, b, c, d$  are real numbers, such that  $H(\alpha(s))$  is a PR function for each  $\alpha(s) \in \Omega$ .

Then, for all SPR0 function  $\nu(s)$  and for all  $\alpha(s) \in \Omega$ , the controller  $C_j(\nu(s))$  stabilizes to the plant  $H(\alpha(s))$  for  $j = 1, \dots, 7$ .

Notice that the plant  $H(s)$  can be unstable and non minimum phase and that the controller  $C_7(\nu(s))$  stabilizes to the plant  $H(\alpha(s))$  for any  $\nu(s) \in \text{SPR0}^*$ .

## 4 Example

We take a plant of the form

$$p_1(s) = \frac{2(s+1)}{s^2+2s-3}$$

and a lead-lag controller

$$c_1(s) = \frac{34.745(s+1.6373)}{s+37.9063},$$

which stabilizes this plant. Let  $\alpha^{-1}(s) = \frac{b-ds}{cs-a}$ , then the closed-loop transfer function is given by

$$H(s) = \frac{c_1(s)p_1(\alpha^{-1}(s))}{1+c_1(s)p_1(\alpha^{-1}(s))}$$

with denominator

$$f(s, a, b, c, d) = A_3 s^3 + A_2 s^2 + A_1 s + A_0$$

and coefficients

$$A_3 = -66.49c^2 - 1.0d^2 + 71.49cd$$

$$A_2 = 2.0bd + 189.59cd - 71.49bc + 132.98ac$$

$$- 5.7077 \times 10^{-2}c^2 - 37.906d^2 - 71.49ad$$

$$A_1 = 0.11415ac - 189.59ad + 75.813bd - 66.49a^2$$

$$- 189.59bc - 1.0b^2 + 71.49ab$$

$$A_0 = 189.59ab - 37.906b^2 - 5.7077 \times 10^{-2}a^2$$

This polynomial is stable if and only if the following inequalities are met:

$$A_0 > 0, \quad A_1 > 0$$

$$A_2 > 0, \quad A_3 > 0$$

$$A_1 A_2 - A_0 A_3 > 0$$

Moreover, we require to meet at least one of the conditions 1., 2., 3., 6., or 7., in Lemma 4. Now by item 1. in Proposition 7, the controllers

$$c_1(\alpha(s)) = \frac{1.73 [(10^4 a + 16373c) s + 10^4 b + 16373.d]}{(500.0a + 18953.c) s + 500.0b + 18953.d}$$

stabilize the plant  $p_1(s)$  for the set of parameters  $a, b, c, d$  that met with the last conditions. For example with  $a, d \in [10.0^{-3}, 8]$ ,  $b, c \in [0, 5]$ , we get the controllers  $c_1(\alpha(s))$  stabilizes  $p_1(s)$ .

## 5 Conclusions

We have characterized the space of parameters  $a, b, c, d$  for which the map  $s \rightarrow \alpha(s)$ , preserves stability for any stable polynomial, mapping stable polynomials in stable polynomials. But also we characterized the space of parameters  $a, b, c, d$  for which the map before mentioned, mapping unstable polynomials in stable polynomials. Like a consequence, we

obtain a dual result, in the sense that the robust stabilization of a plant  $H(s)$  with disturbances nonlinear in its parameters, induced by the substitution of the variable  $s$  by  $\alpha^{-1}(s)$ , with a controller  $C(s)$ , implies the nonfragility of the controller  $C(s)$  under the same class of disturbances, induced by the substitution of the variable  $s$  by  $\alpha(s)$ , in the controller, and the nonfragility of the controller  $C(s)$  under disturbances, induced by the substitution of the variable  $s$  by  $\alpha^{-1}(s)$ , in the controller, implies the robust stabilization of a plant  $H(s)$  with disturbances nonlinear in its parameters, induced by the substitution of the variable  $s$  by  $\alpha(s)$  with a controller  $C(s)$ . In the particular case when  $b = 0$ , the substitution  $\alpha(s)$ , preserves the structure of the controllers type PD/PI/PID and we give some ideas to use later for tuning rules for the derivative part of the controller type PD, and for the proportional and integral part of the controller type PI. Based on the resulting pseudo-parametrization for this type of controller, after of the substitution of the variable  $s$  by  $\alpha(s)$ , taking  $b = 0$ . Finally, we given a result about stabilization based in passivity.

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