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State-feedback Stabilization of Systems with Input Saturation and Measurement Quantization

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Abstract: This note tackles the problem of designing a state-feedback stabilizer of systems not only with input saturation but also with measurement quantization. Since quantization errors are $\mathcal{L}_\infty$-norm bounded, one can describe a ball of origin, rather than an origin, which the state converges. The proposed design method employs two stages: in the first stage, the achievable minimal ball of origin is designed and, in the second stage, the maximal invariant set of initial states is designed.

Key-words: input saturation, measurement quantization

1 Introduction

The problem of stabilizing systems either with input saturation or with quantization error has recently received considerable attention in control societies since such two constraints are frequently encountered in control engineering applications [1]–[10]. Quantization error, similar to system disturbance, occurs in the procedure of converting a real-valued measurement signal into a piecewise constant one taking on a finite set of values. And input saturation occurs in areas where required control actions exceed the threshold of actuator supply.

In [6], T. Hu and Z. Lin described the method of using convex approach for solving saturation nonlinearity. Based on the method, Hu and Lin and Hu *et al.* [7] performed the analysis and design of control systems with the ability of rejecting disturbances bounded in magnitude. In [10], Fang *at el.* extended the work of Hu *et al.* to the systems with disturbances bounded in energy, so-called $\mathcal{L}_2$-disturbances, and produced less conservative results than those of [1] by increasing the degree of freedom for design.

In [4], R. W. Brockett and D. Liberzon proposed a control design methodology in the context of feedback stabilization problem for linear control system with quantization. In [8], D. Liberzon generalized the contribution of [4] by handling the quantized feedback stabilization problem for nonlinear systems and developing results for systems with input quantization, both linear and nonlinear. In [9], H. Ishii and B. A. francis established means to find an upper bound on the data rate to achieve stabilization.

In the case where both the saturation and the quantization error occur at the same time in the stabilization process, one need to incorporate simultaneously the effects of these into the design of a control system. However, to the best of our knowledge, there has been yet no results on designing a state-feedback stabilizer for systems both with input saturation and with quantization error. Thus, in this paper, we shall propose the method of constructing such a stabilizer. To this end, we develop two nested invariant regions such that all trajectories of systems starting in the bigger region approach the smaller one, while no further convergence occurs. Specifically, we obtain the conditions for the existence of two nested invariant regions through two stages: in the first stage, the condition for minimizing a ball of origin is presented, and in the second stage, the condition for maximizing the invariant ellipsoidal set of initial states is presented based on the ball of origin. Since the conditions are formulated in terms of nonlinear matrix inequalities, we shall propose an efficient iterative algorithm involving convex optimization in order to solve these nonlinear matrix inequalities.

The paper follows this outline. Section 2 will describe a problem and define several concepts including a ball of origin, an invariant set of initial states, and so on. Sections 3 and 4 will show how to design the minimal ball of origin and how to construct the maximal invariant set of initial states, respectively. Section 5 will show a simple example for verification of the resulting system.

Notation: Notations in this paper are fairly standard. $\mathcal{L}_\infty$ denotes the space of bounded vector sequences $u(k)$, equipped with the norm $||u||_\infty = \sup_i\{\sup_k |u_i(k)|\}$. The notation $X \geq Y$ and $X > Y$ where X and Y are symmetric matrices means that $X - Y$ is positive semi-definite and positive definite, respectively.

2 Problem Formulation

Consider the following continuous-time system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$
$$u(t) = \text{sat}(K \text{ quan}(x(t))), \quad (2)$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$ denote the state and the control input, respectively, quan$(\cdot)$ denotes a quantization operator, called quantizer, with the sensitivity ϵ and sat$(\cdot)$ denotes a saturation operator with level μ. Here, we assume that the saturation effect of the quantizer can be ignored. In other words, the quantization operator quan$(\cdot)$ yields

$$(\text{quan } (\sigma))_i \triangleq 2\epsilon[\sigma_i/(2\epsilon)], \quad (3)$$

where $[\cdot]$ for a scalar is an integer round-off operator and σ_i denotes the i-th element of σ. And the saturation operator sat$(\cdot)$ yields

$$(\text{sat } (\sigma))_i \triangleq \text{sign}(\sigma_i)\min(\mu, |\sigma_i|), \quad (4)$$

where μ is the saturation level and sign$(\cdot)$ returns the signs of the corresponding argument. In this case, one can find the following relations for two operators:

$$\text{quan } (x) \in \{x + q \mid -\epsilon < q_j \leq \epsilon, \ j \in [1, n]\},$$
$$\text{sat } (z) \in \text{Co}\{\ D_i z + D_i^- w \ \mid \ i \in [1, 2^m]\ \},$$

where $w \in \mathcal{R}^m$ is an auxiliary vector yielding $|w_i| \leq \mu$, ϵ indicates the sensitivity of the quantizer, D_i denotes a diagonal matrix with all possible combinations of "1" and "0" entries, and $D_i^- \triangleq I - D_i$. Using two relations, we can get the following

$$\text{sat}(K\text{quan}(x)) \in \text{Co}\{\ D_i K(x+q) + D_i^- w \ \mid -\epsilon < q_j \leq \epsilon, \ j \in [1, n], \ i \in [1, 2^m]\ \}, \quad (5)$$

where w is a vector yielding $|w_i| \leq \mu$.

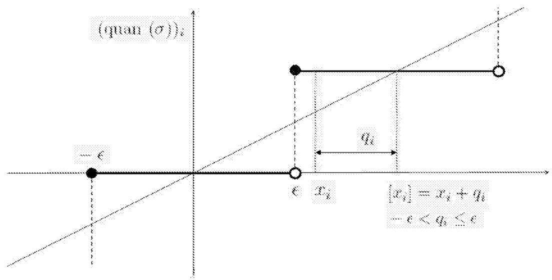

Figure 1: Uniform quantization: $(\text{quan}(x))_i$

The closed-loop system can be described as

$$\dot{x}(t) = Ax(t) + B \text{ sat}(K \text{ quan}(x(t))). \quad (6)$$

Since, the trajectories of the system under the memoryless quantizer (3) can never converge to the origin, we shall obtain two nested invariant regions such that all trajectories of systems starting in the bigger region approach the smaller one, while no further convergence occurs.

3 Minimizing A Ball of Origin

A ball of origin $\mathcal{B}_{P_q}$ is a region where the state remains independently from the quantization errors: for all $t > 0$

$$x(t) \in \mathcal{B}_{P_q} \triangleq \{\ x \in \mathcal{R}^n \mid x^T P_q x < 1\ \}. \quad (7)$$

For Lyapunov function $V(x(t)) = x^T(t)P_q x(t)$,

$$\dot{V} = 2x^T(t)P_q\{Ax(t) + B\ \mathrm{sat}(K_q\ \mathrm{quan}(x(t)))\} < \epsilon^2 \mathrm{Tr}(\Lambda)\{1 - x^T(t)P_q x(t)\}$$

if the following inequality holds that, for all $i \in [1, 2^m]$ and $j \in [1, m]$,

$$\begin{aligned} 0 &> \begin{bmatrix} \mathcal{G}_q + \epsilon^2\mathrm{Tr}(\Lambda)P_q & P_q B D_i K_q \\ K_q^T D_i^T B^T P_q & -\Lambda \end{bmatrix}, \\ 0 &< \begin{bmatrix} P_q & h_{qj}^T \\ h_{qj} & \mu^2 \end{bmatrix}, \\ \mathcal{G}_q &\triangleq (A + BD_iK_q + BD_i^- H_q)^T P_q + \\ &\quad P_q(A + BD_iK_q + BD_i^- H_q), \end{aligned}$$

Λ is a diagonal matrix with positive diagonal entries and $H_q^T = \begin{bmatrix} h_{q1}^T & \cdots & h_{qm}^T \end{bmatrix}$. Let us modify these inequalities as follows:

$$\begin{aligned} 0 &> \begin{bmatrix} \bar{\mathcal{G}}_q + \epsilon^2\mathrm{Tr}(\Lambda)\bar{P}_q & BD_i\bar{K}_q \\ \bar{K}_q^T D_i^T B^T & -\bar{P}_q\Lambda\bar{P}_q \end{bmatrix}, \\ 0 &< \begin{bmatrix} \bar{P}_q & \bar{h}_{qj}^T \\ \bar{h}_{qj} & \mu^2 \end{bmatrix}, \\ \bar{\mathcal{G}}_q &\triangleq \bar{P}_q A^T + \bar{K}_q^T D_i B^T + \bar{H}_q^T D_i^- B^T + \\ &\quad A\bar{P}_q + BD_i\bar{K}_q + BD_i^-\bar{H}_q, \\ \bar{P}_q &= P_q^{-1},\ \bar{K}_q = K_q\bar{P}_q, \\ \bar{H}_q^T &= \bar{P}_q H_q^T = \begin{bmatrix} \bar{h}_{q1}^T & \cdots & \bar{h}_{qm}^T \end{bmatrix}. \end{aligned}$$

Unfortunately, the resulting condition is not convex. Thus, there needs some trick to handle this non-convexity. One of the most attractive relaxation methods is based on iteration, for which we shall use the modified inequalities: for all $i \in [1, 2^m]$ and $j \in [1, m]$

$$0 > \mathcal{L}_1(\bar{P}_q, \bar{K}_q, \bar{H}_q, M, \Lambda, N, i) \triangleq \begin{bmatrix} \bar{\mathcal{G}}_q + \epsilon^2\mathrm{Tr}(\Lambda)M & BD_i\bar{K}_q & 0 \\ \bar{K}_q^T D_i^T B^T & -N\bar{P}_q - \bar{P}_q N^T & N \\ 0 & N^T & -\Lambda \end{bmatrix}, \quad (8)$$

$$0 < \mathcal{L}_2(\bar{P}_q, \bar{h}_{qj}, j) \triangleq \begin{bmatrix} \bar{P}_q & \bar{h}_{qj}^T \\ \bar{h}_{qj} & \mu^2 \end{bmatrix}, \quad (9)$$

$$0 < \mathcal{L}_3(\bar{P}_q, M) \triangleq M - \bar{P}_q. \quad (10)$$

Proposition 3.1 Procedure to minimize the ball of origin: ***(Initialization)*** *Determine the initial values of N, Λ, and $\bar{P}_q$ yielding (8)-(10): put $N = I$ and $\Lambda = I$; then*

(i) Minimize α subject to $\alpha I > \mathcal{L}_1$, $0 < \mathcal{L}_2$ and $0 < \mathcal{L}_3$ for fixed N and Λ, and then fix $\bar{P}_q$ and M from the feasible entries of the procedure above.

(ii) Minimize α subject to $\alpha I > \mathcal{L}_1$ and $0 < \mathcal{L}_2$ for fixed $\bar{P}_q$ and M.

(ii) If $\alpha \geq 0$, replace the values N_{pre} and Λ_{pre} with the entries computed above and repeat the procedure. Otherwise, put $N_{init} = N$, $\Lambda_{init} = \Lambda$, and $\bar{P}_{q,init} = \bar{P}_q$ from the previous procedure and skip to the following 3-phase procedure.

(3-Phase Iteration) *Put $N = N_{init}$, $\Lambda = \Lambda_{init}$, and $\bar{P}_{q,pre} = \bar{P}_{q,init}$.*

(i) Minimize the trace of $\bar{P}_q$ subject to (8)-(10) for fixed N and Λ.

(ii) Minimize the trace of M subject to (8)-(10) for fixed $\bar{P}_q$ and Λ.

(iii) Minimize the trace of $\bar{P}_q$ subject to (8)-(10) for fixed N and M.

(iv) If $|\bar{P}_q - \bar{P}_{q,pre}| \leq \rho \simeq 0$, stop the procedure; otherwise, assign $\bar{P}_{q,pre} = \bar{P}_q$ and repeat the procedure.

(Controller Construction) $K_q = \bar{K}_q\bar{P}_q^{-1}$. ∎

4 Maximizing an Invariant Set of Initial States

We consider an invariant set of initial states, $\mathcal{E}_P$, such that the state $x(t) \in \mathcal{E}_P$ converges into the ball of origin $\mathcal{B}_{P_q}$ as t goes to infinity:

$$\mathcal{E}_P \triangleq \{ x \mid x^T P x < 1 \}, \tag{11}$$

where $\mathcal{E}_P$ includes $\mathcal{B}_{P_q}$, *i.e.* $P \leq P_q$. We shall use a Lyapunov function $V(x(t)) = x^T(t)Px(t)$, whose derivative satisfies the following condition

$$\begin{aligned}\dot{V} &= 2x^T(t)P\{Ax(t) + B\ \mathrm{sat}(K\ \mathrm{quan}(x(t)))\} \\ &< \epsilon^2 \mathrm{Tr}(\Lambda)(1 - x^T(t)P_q x(t)).\end{aligned} \tag{12}$$

This condition can be achieved via the following inequality: for all $i \in [1, 2^m]$ and $j \in [1, m]$

$$\begin{aligned} 0 &> \begin{bmatrix} \mathcal{G} + \epsilon^2 \mathrm{Tr}(\Lambda) P_q & PBD_iK \\ K^T D_i^T B^T P & -\Lambda \end{bmatrix}, \\ 0 &< \begin{bmatrix} P & h_j^T \\ h_j & \mu^2 \end{bmatrix}, \ P \leq P_q, \\ \mathcal{G} &\triangleq (A + BD_iK + BD_i^- H)^T P + \\ &\quad P(A + BD_iK + BD_i^- H), \end{aligned}$$

Let us modify the following matrix inequalities: for all $i \in [1, 2^m]$ and $j \in [1, m]$

$$\begin{aligned} 0 &> \begin{bmatrix} \bar{\mathcal{G}} + \epsilon^2 \mathrm{Tr}(\Lambda) \bar{P} P_q \bar{P} & BD_i\bar{K} \\ \bar{K}^T D_i B^T & -\bar{P}\Lambda\bar{P} \end{bmatrix}, \\ 0 &< \begin{bmatrix} \bar{P} & \bar{h}_j^T \\ \bar{h}_j & \mu^2 \end{bmatrix}, \ \bar{P} \geq \bar{P}_q, \\ \bar{\mathcal{G}} &\triangleq \bar{P}A^T + \bar{K}^T D_i B^T + \bar{H}^T D_i^- B^T + \\ &\quad A\bar{P} + BD_i\bar{K} + BD_i^- \bar{H}, \\ \bar{P} &= P^{-1}, \bar{K} = K\bar{P}, \\ \bar{H}^T &= \bar{P}H^T = \begin{bmatrix} \bar{h}_1^T & \cdots & \bar{h}_m^T \end{bmatrix}. \end{aligned}$$

Since these conditions are also non-convex, we use the same iterative method as Proposition 3.1 to maximize the invariant set of initial states, for which we shall use the modified inequalities: for all $i \in [1, 2^m]$ and $j \in [1, m]$

$$0 > \mathcal{L}_4(\bar{P}, \bar{K}, \bar{H}, \Lambda, M, N, i) \triangleq \begin{bmatrix} \bar{\mathcal{G}} + \epsilon^2 \mathrm{Tr}(\Lambda) M & BD_i\bar{K} & 0 \\ \bar{K}^T D_i B^T & -N\bar{P} - \bar{P}N^T & N \\ 0 & N^T & -\Lambda \end{bmatrix}, \tag{13}$$

$$0 < \mathcal{L}_5(\bar{P}, \bar{h}_j) \triangleq \begin{bmatrix} \bar{P} & \bar{h}_j^T \\ \bar{h}_j & \mu^2 \end{bmatrix}, \tag{14}$$

$$0 \leq \mathcal{L}_6(\bar{P}) \triangleq \bar{P} - \bar{P}_q, \tag{15}$$

$$0 < \mathcal{L}_7(M, \bar{P}) \triangleq \begin{bmatrix} M & \bar{P} \\ \bar{P} & \bar{P}_q \end{bmatrix}. \tag{16}$$

Proposition 4.1 Procedure to maximize the invariant set of initial states: *(Initialization)* *Determine the initial values of N, Λ, and $\bar{P}$ yielding (13)-(16): put $N = I$ and $\Lambda = I$; then*

(i) Minimize α subject to $\alpha I > \mathcal{L}_4$, $0 < \mathcal{L}_5$, $0 < \mathcal{L}_6$, and $0 < \mathcal{L}_7$ for fixed N and Λ, and then fix $\bar{P}$ and M from the feasible entries of the procedure above.

(ii) Minimize α subject to $\alpha I > \mathcal{L}_4$, and $0 < \mathcal{L}_5$ for fixed $\bar{P}$ and M.

(iii) If $\alpha \geq 0$, replace the values N_{pre} and Λ_{pre} with the entries computed above and repeat the procedure. Otherwise, put $N_{init} = N$, $\Lambda_{init} = \Lambda$, and $\bar{P}_{init} = \bar{P}$ from the previous procedure and skip to the following 3-phase procedure.

(3-Phase Iteration) *Put $N = N_{init}$, $\Lambda = \Lambda_{init}$, and $\bar{P}_{pre} = \bar{P}_{init}$.*

(i) Minimize the trace of $\bar{P}$ subject to (13)-(16) for fixed N and Λ.

(ii) Minimize the trace of M subject to (13)-(16) for fixed $\bar{P}$ and Λ.

(iii) Minimize the trace of $\bar{P}$ subject to (13)-(16) for fixed N and M.

(iv) If $|\bar{P} - \bar{P}_{pre}| \leq \rho \simeq 0$, stop the procedure; otherwise, assign $\bar{P}_{pre} = \bar{P}$ and repeat the procedure.

(Controller Construction) $K = \bar{K}\bar{P}^{-1}$. ∎

Proposition 4.2 Two-stage Control: *For the system with some initial states in $\mathcal{E}_P$, apply for $u = sat(K quan(x(t)))$. Once the state $x(t)$ reaches into the ball of origin $\mathcal{B}_{P_q}$, replace the control with $u = sat(K_q quan(x(t)))$. This control strategy guarantees the asymptotical convergence of the state in $\mathcal{E}_P$ to the ball of origin $\mathcal{B}_{P_q}$ and also the stay in $\mathcal{B}_{P_q}$ after arriving.* ∎

5 Numerical Example

To demonstrate the performance of the proposed algorithm, let us consider the following system:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \tag{17}$$

Assume that $||x||_\infty \leq 10$ and the measured

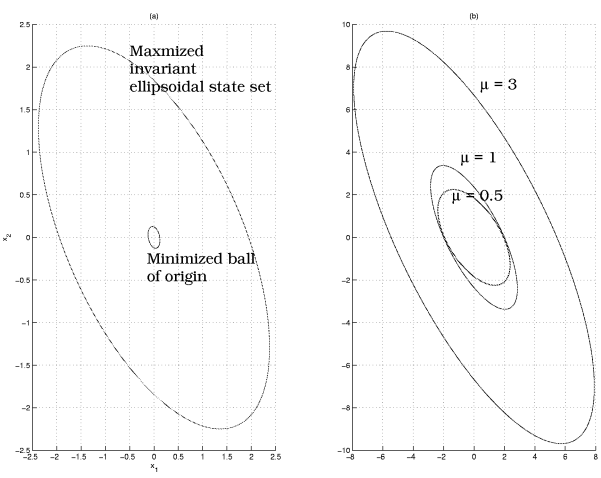

Figure 2: (a) Minimal ball of origin $\mathcal{B}_{P_q}$ and maximal invariant set of initial states $\mathcal{E}_P$, (b) Maximal invariant sets of initial states for $\mu = 0.5, 1, 3$.

state $x(t)$ is quantized by 8-bit digital signals. From the assumption, the quantization level ϵ is given as $\epsilon = 10/2^8$. For the control constraint $\mu = 0.5$, Fig. 2-(a) shows two nested invariant regions obtained from the proposed iterative algorithm. Fig. 2-(b) shows that the size of the invariant set $\mathcal{E}_P$ increases as the control constraint μ increases.

6 Conclusion

In this paper, we proposed the method of constructing state-feedback stabilization problems for systems with input saturation and measurement quantization. The conditions for the existence of two nested invariant regions such that all trajectories of systems starting in the bigger region approach the smaller one, while no further convergence occurs, are formulated in terms of nonlinear matrix inequalities, and then an iterative algorithm involving convex optimization is proposed in order to solve these nonlinear matrix inequalities.

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