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Control Design for Uncertain Systems Based on Integral-Type Sliding Surface

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Abstract: - An advanced robust sliding mode control scheme is proposed for linear systems. First, based on LMI technique the positive definite matrix which is key to construct the sliding surface is found , then a global integral-type sliding surface is constructed for the systems in the presence of both matched and unmatched uncertainties, then the global sliding mode controller is constructed for systems both with matched and unmatched uncertainties. In terms of LMIs the sliding surface is easy to design especially for even large-scale systems in the computational aspect. Furthermore the controlled system during ideal sliding mode completely nullifies matched uncertainties and inherits the same properties as those of the controlled nominal system in the absence of uncertainties. In addition, since the reaching phase is eliminated, the controlled system is more robust against perturbations than the other variable structure control system with reaching phase.

Key words -Sliding model, variable structure system control, LMIs, nominal system, unmatched uncertainties.

1 Introduction

Variable structure system control, as one of the most active research areas of control theory and one of the powerful practical tools, has been studied for many decades[1-9]. Sliding mode controller is constructed in order to keep controlled systems to a given constraint surface and also make the systems insensitive to some certain external and internal disturbances. The basic idea of sliding mode control is as follows. Choose a sliding surface; then use the sliding mode controller to drive the state outside into the surface; finally, using equivalent control to render the state in the sliding mode along the surface to the desired equilibrium.

Many approaches have been proposed for the design of the sliding surface-these include pole placement, eigenstructure assignment and optimal quadratic methods. More recently linear matrix inequality (LMI) methods have been explored[2-9]. Choi[2] developed a new design method of linear sliding surface which is linear to the state. This method has some advantages over traditional design methods and offers some additional design flexibility. Since LMI problems can be easily determined whether they are feasible or not, and if it is, they can be solved very efficiently by using powerful algorithms. This approach has advantage in the computational aspect and therefore sliding surface for even large-scale systems can be easily computed. Using this method, the emphasis in the work of Choi[2,7] and Gouaisbaut[6] has been largely the design of sliding surface so that the sliding mode is robust to unmatched uncertainty. In [3,4] Choi considered the problem of designing a sliding surface for a class of uncertain systems with mismatched uncertainties. The mismatched uncertainties is the form $DF(x,t)Ex(t)$, where $F(x,t)$ is unknown but bounded as $\|F(x,t)\| < 1$ for all $(x,t) \in R^n \times R$, and D and E are known matrices of appropriate dimensions. In order to improve the robustness of the systems, in this paper we design a integral-type surface based on LMI. The systems considered in the paper are wider than the systems considered in [3,4] and the sliding mode controller is global sliding mode controller, therefore the reaching intervals is eliminated and the

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robustness of the closed systems is improved.

2 Problem formulation

Let the system to be controlled be represented by the following differential equation

$$\dot{x}(t) = Ax(t) + B[(I + Y(x,t))u(t) + h(x,t)] + w(x,p,t), \qquad 1$$

where $x \in R^n$ is a vector of measurable states, $u \in R^m$ is a vector of control inputs, and $A \in R^{n\times n}$ is the system-characteristics matrix, $B \in R^{n\times m}$ is the input matrix with full rank $r(B) = m$. $Y(x,t)$ and $h(x,t)$ represent the matched uncertainties and $w(x,p,t)$ represents the unmatched uncertainties, where $p \in S$, and $S \in R^m$ is some prescribed compact set.

We will assume the following to be valid :

1) The pair (A,B) is stabilizable.

2) The matched uncertainties $Y(x,t)$ are bounded in Euclidean norm as

$$\|Y(x,t)\| < 1-\varepsilon, \quad 0<\varepsilon<1$$

and $\|h(x,t)\| \le \beta(x,t)$ where $\beta(x,t)$ is a known continuous positive scalar-valued function. There exist functions $\alpha(x,t)$ such that

$$\|w(x,p,t)\| \le \alpha(x,t)$$

and $\alpha(x,t)$ is a known continuous positive scalar-valued function.

The systems 1 are more general kind systems than those of [3,4].

3 Design of sliding surface

In order to design the sliding controller of the system (1), we first consider the nominal form of system (1) as following

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad 2$$

the following result is very useful.

Lemma 1[7] If the pair (A,B) is stabilizable, then the following LMIs have a solution matrix X :

$$X > 0, \ \tilde{B}^T(AX + XA^T)\tilde{B} < 0.$$

Where $\tilde{B} \in R^{n\times(n-m)}$ is the orthogonal complement of the input matrix B that is $\tilde{B}$ satisfying

$$B^T\tilde{B} = 0, \quad \tilde{B}^T\tilde{B} = I.$$

Consider the nominal systems (2), the sliding surface, which is linear with respect to the state x is given by

$$\sigma_1(x) = Sx = B^T X^{-1}x = 0. \qquad 3$$

The sliding mode controller is chosen as

$$u(t) = -(SB)^{-1}(SA - \Phi S)x, \qquad 4$$

where Φ is chosen negative definite matrix.

We can establish the following.

Theorem 1 Consider the nominal systems (2), the sliding surface is chosen as (3) and the sliding mode controller is chosen as (4), then the closed-loop systems restricted to the sliding surface is stable and the resulting $n-m$ reduced-order dynamics is given by

$$\dot{x}(t) = [A - B(SB)^{-1}(SA - \Phi S)]x(t), \qquad 5$$

or

$$\tilde{B}^T\dot{x}(t) = \tilde{B}^T AX\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}\tilde{B}^T x(t). \qquad 5'$$

Proof Let $V(x) = \frac{1}{2}\sigma_1^T\sigma_1$ then

$$\begin{aligned}\dot{V} &= \sigma_1^T\dot{\sigma}_1 = \sigma_1^T[S(Ax(t) + Bu(t))] \\ &= \sigma_1^T[SA - SB(SB)^{-1}(SA - \Phi S)]x(t) \\ &= \sigma_1^T\Phi Sx(t) = \sigma_1^T\Phi\sigma_1 < 0\end{aligned}$$

so the reachability condition is satisfied.

Define a transformation matrix as follows

$$M = \begin{bmatrix}\tilde{B}^T \\ B^T X^{-1}\end{bmatrix} = \begin{bmatrix}\tilde{B}^T \\ S\end{bmatrix}$$

then $M^{-1} = [X\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}, \quad B(SB)^{-1}]$.

Let $z(t) = Mx(t)$ then $\begin{bmatrix} z_1(t) \\ z_2(t)\end{bmatrix} = \begin{bmatrix}\tilde{B}^T x \\ \sigma_1(x)\end{bmatrix}$

where $z_1(t) \in R^{n-m}$, $z_2(t) \in R^m$. system 2 is transformed into follows

$$\dot{z}(t) = MAM^{-1}z(t) + MBu(t) \qquad (6)$$

Now using (4) in equation (6) gives

$$\dot{z}(t) = \begin{bmatrix}\tilde{B}^T AX\tilde{B}(\tilde{B}^T X\tilde{B})^{-1} & \tilde{B}^T AB(SB)^{-1} \\ 0 & \Phi\end{bmatrix}\cdot\begin{bmatrix} z_1 \\ z_2\end{bmatrix}$$

noted that on the sliding surface $z_2(t) = \sigma_1 = 0$, then the system dynamics is governed by equation $\tilde{B}^T\dot{x}(t) = \tilde{B}^T AX\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}\tilde{B}^T x(t)$ which is asymptotically stable.

In order to improve the robust quality of the closed-loop systems of (1), it is proposed the following integral type sliding surface

$$\sigma(x) = Sx(t) - Sx(t_0) - \int_{t_0}^{t}\Phi Sx(\tau)d\tau. \qquad 7$$

Obviously the nice property $\sigma(x(t_0)) = 0$ is achieved, such that the reaching phase is eliminated. Since the sliding mode exists from the very beginning, the system is more robust against perturbations than the other sliding mode control systems with reaching phase.

4 Construction of controllers

First we consider systems (1) only with matched uncertainties. The systems is prescribed as follows

$$\dot{x}(t) = Ax(t) + B[(I + Y(x,t))u(t) + h(x,t)], \quad (8)$$

using the sliding surface 7 and the controller is designed as follows

$$u(t) = \begin{cases} -(SB)^{-1}(SA - \Phi S)x(t) \\ -\rho(x,t)(SB)^T\sigma \big/ \left\|(SB)^T\sigma\right\|, \text{if } \sigma \neq 0 \\ -(SB)^{-1}(SA - \Phi S)x(t), \text{if } \sigma = 0 \end{cases} \quad (9)$$

Where

$$\rho(x,t) > \varepsilon^{-1}\{(1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\| + \beta(x,t)\}.$$

Theorem 2 Consider system (8), if the sliding surface is chosen as (7) and the sliding mode controller is (9), then the closed-loop system is asymptotically stable and the closed-loop dynamics restricted on the surface is

$$\dot{x}(t) = [A - B(SB)^{-1}(SA - \Phi S)]x(t). \quad 10$$

Proof Differentiating the sliding surface $\sigma(x)$ with respect to time using (8) and (9) one obtains

$$\begin{aligned}\dot{\sigma}(x) &= S\dot{x}(t) - \Phi Sx(t) \\ &= SB[(I + Y(x,t))u(t) + h(x,t) \\ &\quad + (SB)^{-1}(SA - \Phi S)x(t)] \\ &= SB(I + Y(x,t))u(t) + SBh(x,t) \\ &\quad + (SA - \Phi S)x(t).\end{aligned} \quad 11$$

Let $V(x) = \frac{1}{2}\sigma^T\sigma$ then

$$\begin{aligned}\dot{V} = \sigma^T\dot{\sigma} &= \sigma^T SB[(I + Y(x,t))u(t) + h(x,t) \\ &\quad + (SB)^{-1}(SA - \Phi S)x(t)].\end{aligned}$$

If $\sigma \neq 0$, substituting sliding mode controller (9) into the previous equation one obtains

$$\begin{aligned}\dot{V} &= \sigma^T SB\{-Y(x,t)(SB)^{-1}(SA - \Phi S)x(t) \\ &\quad - \rho(x,t)Y(x,t)\frac{(SB)^T\sigma}{\left\|(SB)^T\sigma\right\|} - \rho(x,t)\frac{(SB)^T\sigma}{\left\|(SB)^T\sigma\right\|} \\ &\quad + h(x,t)\} \\ &\leq \{[(1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\|] \\ &\quad + (1-\varepsilon)\rho(x,t) - \rho(x,t) + h(x,t)\}\left\|(SB)^T\sigma\right\| \\ &\leq -\{\varepsilon\rho(x,t) - (1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\| \\ &\quad - \beta(x,t)\}\left\|(SB)^T\sigma\right\| < 0,\end{aligned}$$

thus $\sigma = 0$, $\forall t \in [t_0, +\infty)$. In the sliding mode, $\dot{\sigma} = 0$, thus in the sliding mode the controller is

$$\begin{aligned}u(t) &= [I + Y(x,t)]^{-1}[-h(x,t) \\ &\quad - (SB)^{-1}(SA - \Phi S)x(t)]\end{aligned} \quad 12$$

Substituting (12) into (8) one obtains (10).

In the sliding mode, the matched uncertainties $Y(x,t)$, $h(x,t)$ are completely nullified. Moreover, under the sliding surface (7), the closed-loop dynamics of systems (8) in the sliding mode mimics the nominal systems (2) under the nominal controller (4).

Now we consider the use of integral type sliding surface (7) for a general class of unmatched uncertainties systems with form (1). The sliding controller is designed as

$$u(t) = \begin{cases} -(SB)^{-1}(SA - \Phi S)x(t) \\ -\rho_u(x,t)(SB)^T\sigma \big/ \left\|(SB)^T\sigma\right\|, \text{if } \sigma \neq 0, \\ -(SB)^{-1}(SA - \Phi S)x(t), \text{if } \sigma = 0, \end{cases} \quad 13$$

where

$$\begin{aligned}\rho_u(x,t) &> \varepsilon^{-1}[(1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\| \\ &\quad + \beta(x,t) + \alpha(x,t)\left\|(SB)^{-1}S\right\|].\end{aligned}$$

Theorem 3 Consider system (1), if the sliding surface is chosen as (7) and the sliding mode controller is (13), then the closed-loop system is asymptotically stable and the closed-loop dynamics restricted on the surface is

$$\begin{aligned}\dot{x}(t) &= [A - B(SB)^{-1}(SA - \Phi S)]x(t) \\ &\quad + [I - B(SB)^{-1}S]w(x,p,t)\end{aligned} \quad 14$$

Proof Differentiating the sliding surface $\sigma(x)$ with respect to time using (1) and (13) one obtains

$$\begin{aligned}\dot{\sigma}(x) &= S\dot{x}(t) - \Phi Sx(t) \\ &= SB(I + Y(x,t))u(t) + SBh(x,t) \\ &\quad + Sw(x,p,t) + (SA - \Phi S)x(t).\end{aligned} \quad 15$$

Let $V(x) = \frac{1}{2}\sigma^T\sigma$ then

$$\dot{V} = \sigma^T \dot{\sigma}$$
$$= \sigma^T SB[(I + Y(x,t))u(t) + h(x,t) +$$
$$(SB)^{-1}(SA - \Phi S)x(t) + (SB)^{-1} Sw(x, p, t)].$$

If $\sigma \neq 0$, substituting sliding mode controller (13) into the previous equation one obtains

$$\dot{V} = \sigma^T SB\{-Y(x,t)(SB)^{-1}(SA - \Phi S)x(t)$$
$$-\rho_u(x,t)Y(x,t)\frac{(SB)^T\sigma}{\left\|(SB)^T\sigma\right\|} - \rho_u(x,t)\frac{(SB)^T\sigma}{\left\|(SB)^T\sigma\right\|}$$
$$+ h(x,t) + (SB)^{-1}Sw(x,p,t)\}$$
$$\leq \{[(1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\|]$$
$$+ (1-\varepsilon)\rho_u(x,t) - \rho_u(x,t) + h(x,t)$$
$$+ \left\|(SB)^{-1}S\right\|\alpha(x,t)\}\left\|(SB)^T\sigma\right\|$$
$$\leq -\{\varepsilon\rho_u(x,t) - (1-\varepsilon)\left\|(SB)^{-1}(SA - \Phi S)x(t)\right\|$$
$$-\beta(x,t) - \left\|(SB)^{-1}S\right\|\alpha(x,t)\}\left\|(SB)^T\sigma\right\| < 0.$$

In the sliding mode, $\dot{\sigma} = 0$, thus in the sliding mode the controller is

$$u(t) = [I + Y(x,t)]^{-1}[-h(x,t)$$
$$-(SB)^{-1}(SA - \Phi S)x(t) - (SB)^{-1}Sw(x,p,t)] \quad 16$$

submitting (16) into (1) we get (14).

5 Conclusions

In this paper, a global integral-type sliding surface is proposed for the system in the presence of both matched and unmatched uncertainties. In terms of LMIs the sliding surface is easy to design especially for even large-scale systems in the computational aspect. Furthermore the controlled system during ideal sliding mode completely nullifies matched uncertainties and inherits the same properties as those of the controlled nominal system in the absence of uncertainties and the nature and the size of the of equivalent unmatched uncertainties.

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