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Point values Hermite multiresolution for non-smooth noisy signals II

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Abstract: This paper is devoted to the construction of a nonlinear interpolation in order to compute adaptively derivatives from signal discrete data. Using these derivatives a multiresolution based on Hermite interpolation is performed. The way in which the derivatives are approximated is crucial.

Key Words: Multiresolution, Hermite interpolation, derivatives, nonlinear reconstructions.

1 Introduction

Harten's multiresolution representation of data is an useful tool for data compression. Given a finite sequence f^L, which represents sampling of weighted-averages of a function $f(x)$ at the finest resolution level L, multiresolution algorithms connect it with its multiscale representation

$$\{f^0, d^1, d^2, \ldots, d^L\},$$

where the f^0 corresponds to the sampling at the coarsest resolution level and each sequence d^k represents the intermediate details which are necessary to recover f^k from f^{k-1}.

In [6] an efficient multiresolution framework based on Hermite interpolation is presented. The function and derivative point-values are used. In practice, we usually have to approximate these derivatives. In [3], it is presented an algorithm based on ENO (essential non oscillatory) [4] interpolation in order to approximate accurately the derivatives of piecewise smooth functions. In [1] we show that this approach is not good for noisy signals and we present an algorithm based on a fifth order generalization of the PPH (piecewise polynomial harmonic) reconstruction [2]. This scheme is based in a modification of Lagrange's interpolations using har-

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monic means. The difference between the classical arithmetic means that appear in Lagrange's interpolation and the proposed harmonic means are of order two. In order to obtain the desired fifth order accuracy, we consider new "phantoms" grid points. However, the use of these new points can present numerical stability problems. Moreover, the algorithm seems too technical. In the present paper, we use a new class of means presented in [5]. With these means we can obtain a new adaptive algorithm with the desired accuracy but without using any "phantoms" grid point.

The paper is organized as follows: In section 2 we briefly review the multiresolution based on Hermite interpolation. In section 3 the new nonlinear interpolation is presented.

2 The Interpolatory Multiresolution Setting

Let us consider a set of nested grids:

$$X^k = \{x_j^k\}_{j=0}^{J_k}, \quad x_j^k = jh_k, \quad h_k = 2^{-k}/J_0,$$

where $J_k = 2^k J_0$, J_0 some fixed integer. Consider the point-value discretization

$$D_k : \begin{cases} C([0,1]) & \to V^k \\ f & \mapsto f^k = (f_j^k)_{j=0}^{J_k} \end{cases} \tag{1}$$

where V^k is the space of real sequences of length J_k+1. A reconstruction procedure for this discretization operator is any operator R_k such that

$$R_k : V^k \to C([0,1]); \quad D_k R_k f^k = f^k, \tag{2}$$

which means that

$$(R_k f^k)(x_j^k) = f_j^k = f(x_j^k). \tag{3}$$

In other words, $(R_k f^k)(x)$ is a continuous function that interpolates the data f^k on X^k.

2.1 Multiresolution based on Hermite interpolation

Warming and Beam [6] generalize the above framework in order to incorporate the information coming from the derivatives and to use Hermite interpolation. We briefly review this setting.

We define lower resolution grids $x^k = \{x_j^k\}_{j=0}^{J_k}$where $k = 1, ..., L$ by dyadic coarsening: $x_j^{k-1} = x_{2j}^k$, $j = 0, ..., J_{k-1} = \frac{J_k}{2}$

We suppose that the function $f \in \mathcal{F}$ and its derivatives are given on the mesh X^m. We define:

$$f_j^k = f(x_j^k),$$

$$g_j^k = h_k f'(x_j^k).$$

We have to decimate and predict the two sets of data f and g. The direct mutiresolution algorithm, according to [6], is as follows:

$$\begin{cases} \text{Do} \quad k = 1, \ldots, L \\ f_j^{k-1} = f_{2j}^k, \\ g_j^{k-1} = 2g_{2j}^k, \\ (d_f)_j^k = f_{2j-1}^k - \frac{1}{2}\left(f_{2j}^k + f_{2j-2}^k \right) + \frac{1}{4}\left(g_{2j}^k - g_{2j-2}^k \right), \\ (d_g)_j^k = g_{2j-1}^k - \frac{3}{4}\left(f_{2j}^k - f_{2j-2}^k \right) + \frac{1}{4}\left(g_{2j}^k + g_{2j-2}^k \right). \end{cases} \tag{4}$$

3 Computing derivatives

As in most practical situations only function point-values are given, we shall compute the

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approximate values at the nodes of the finest grid X^m with the appropriate accuracy and then apply the two algorithms of multiresolution. The way in which these approximations are computed turns out to be crucial in terms of accuracy and data compression.

Linear reconstruction techniques associated to large supports are affected by the presence of singularities. We showed in [1] that the option to obtain adaptation near singularities such as ENO schemes have problems in the presence of noise. In [1], in order to obtain the desired fifth order accuracy using a PPH-type reconstruction [2] and improve the behavior of ENO schemes, we propose a "phantom" strategy. With this strategy some numerical stability problems can appear.

In the present paper, we present a new nonlinear reconstruction technique associated to a stencil of five points. We expect that its "locality" (with centered stencil) leads to improvements specially when noise is presented. In this algorithm, it is not necessary the "phantom" strategy and we have not numerical stability problems.

3.1 The new non linear reconstruction technique

In this section we present a fifth order non linear and data dependent interpolation technique, using similar ideas than in the PPH reconstruction but using a different mean: the 3-power mean [5]. Notice that the original PPH reconstruction is based on the harmonic (2-power) mean.

Let x, y be two real numbers with the same sign. We define the p-power mean as

$$power_p(x,y) = min(x,y) \cdot \Big(1 + |\frac{y-x}{y+x}| + \cdots + |\frac{y-x}{y+x}|^p\Big)$$

The infinity series converge to $(x+y)/2$.

We will use two properties of these means:

a)

$$\frac{x+y}{2} - power_p(x,y) = O(|x-y|^p)$$

(related with the accuracy in smooth regions).

b)

$$|power_p(x,y)| \leq p\min(|x|,|y|)$$

(related with the adaptation to the singularities).

Starting from the set of points $f_{j-2}, f_{j-1}, f_j, f_{j+1}, f_{j+2}$ and the polynomial

$$P(x) = a_0 + a_1(x-x_j) + a_2(x-x_j)^2 + a_3(x-x_j)^3 + a_4(x-x_j)^4,$$

we have :

$$a_1 = \frac{f_{j-2} - 8f_{j-1} + 8f_{j+1} - f_{j+2}}{12h}.$$

Let us introduce the divided differences defined by:

$$\begin{aligned} d_{j-r} &= \frac{f_{j-r+1} - f_{j-r}}{h}; \quad r = -1, 0, 1, 2 \\ e_{j-r} &= \frac{d_{j-r+1} - d_{j-r}}{h}; \quad r = 0, 1, 2. \end{aligned}$$

In this case, any "phantom" points appear.

Firstly, we assume that $| e_j |>| e_{j-2} |$. This indicates the presence of a possible singularity at a point $x_d \in [x_j, x_{j+2}]$.

From

$$a_1 = \frac{-2\frac{e_j+e_{j-2}}{2} + 12\frac{e_{j-1}+e_{j-2}}{2} - 4e_{j-2}}{12}h + d_{j-1}.$$

We define the new modified value of a_1 as

$$\begin{aligned}\tilde{a}_1 &= \frac{-2H(e_j, e_{j-2}) + 12H(e_{j-1}, e_{j-2}) - 4e_{j-2}}{12}h \\ &+ d_{j-1},\end{aligned}$$

where

$$H(x,y) = \begin{cases} power_3(x,y) & xy > 0 \\ 0 & xy \leq 0 \end{cases} \quad (5)$$

In smooth regions, as soon as $e_{j-1}e_{j-2} > 0$ and $e_j e_{j-2} > 0$, then $a_1 - \tilde{a}_1 = O(h^4)$, since the difference between the $3-$power mean and the original arithmetic mean is $O(h^3)$. Then, the fifth order of accuracy for the original signal.

On the other hand, in the presence of a discontinuity $x_d \in [x_j, x_{j+2}]$, we have that $\mid e_j \mid= O(1/h^2)$ and $\mid\mid e_{j-2} \mid= O(1)$, but since, $|power_3(x,y)| \leq 3\min(|x|,|y|)$, then $\tilde{a}_1$ remains $O(1)$, obtaining the desired adaptation.

Finally, we notice you that the modifications in the case of $|e_j| \leq |e_{j-2}|$ are similar for the symmetry.

The nonlinearity appears in two steps. Firstly, a selection procedure that using divided differences marks points corresponding to possible singularities. Secondly, for these points a nonlinear interpolation is performed introducing a modification of the Lagrange interpolation using p-power means. With these means we obtain the desired accuracy but without the "phantom" strategy and without possible numerical stability problems. Finally, the new algorithm seems lees elaborate.

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