Proceedings of the 5th WSEAS Int. Conf. on Signal Processing, Robotics and Automation, Madrid, Spain, February 15-17, 2006 (pp16-21)

On the stability of a class of nonlinear Interpolatory Wavelet-Packets schemes

S. AMAT, S. BUSQUIER, A. ESCUDERO, J. LACOMBE AND J. LIANDRAT
Departamento de Matemática Aplicada y Estadística.
Universidad Politécnica de Cartagena.
Paseo Alfonso XIII,52. 30203 Cartagena(Murcia).
Spain.

Ecole Généraliste d'Ingénieurs de Marseille (EGIM).
Laboratoire d'Analyse Topologie et Probabilites (LATP).

Abstract: This paper is devoted to the stability of a class of wavelets-packets multiresolution algorithms. These schemes allows to establish a contractivity property that leads to the stability.

Key Words: Wavelet-Packets, interpolation, nonlinear schemes, stability, contractivity.

1 Introduction

Multiresolution representations of data, such as wavelet-packets decompositions, are a powerful tool in several areas of application. In such applications, one typically exploits the ability of these representations to approximate the input data with a high accuracy by a very small set of coefficients.

The concept of wavelet-packets has been introduced by Coifman et al. [7], [8], [9] as a generalization of wavelet bases. It relies on the definition of a library of bases. The best base is chosen so as to minimize some given entropy attached to the coefficients in each base of the library. This idea can be made precise as well as generalized to all multiresolution with a tree structure. This flexibility of choosing the decomposition of the signal is well adapted for applications.

In practical applications, as we said before, a discrete sequence s_0^L is encoded to produce a multiscale representation of its information contents, $(s_0^0, s_1^0, \ldots, s_{2^L-1}^0)$; this representation is then processed and the end result of this step is a modified multiscale representation $(\hat{s}_0^0, \hat{s}_1^0, \ldots, \hat{s}_{2^L-1}^0)$ which is *close* to the original one, i.e. such that (in some norm)

$$||\hat{s}_i^0 - s_i^0|| \leq \epsilon$$

After decoding the processed representation, we obtain a discrete set $\hat{s}_0^L$ which is expected to be *close* to the original discrete set s_0^L. In order for this to be true, some form of stability is needed, i.e. we must require that

$$||\hat{s}_0^L - s_0^L|| \leq C\epsilon$$

where C is a positive constant.

In the last years, several ways to improve the classical linear multiresolutions of wavelet type have lead to nonlinear multiresolutions. This nonlinear nature is a source of difficulty for the proofs of convergence and stability.

The aim of this paper is to establish the stability of a family of both linear and nonlinear wavelet-packets multiresolutions. We obtain explicit error bounds.

The paper is organized as follows: In section 2 we recall briefly the interpolatory multiresolution Harten's framework. We introduce the wavelet-paquets schemes in 3. In section 4 we establish a contractivity property and we give the stability.

2 Harten's framework

The discrete multiresolution framework introduced by Harten is based on two operators: decimation and prediction.

$$D_k^{k-1} : V^k \to V^{k-1}, \tag{1}$$

$$P_{k-1}^k : V^{k-1} \to V^k. \tag{2}$$

From a set of discrete data $f^k = (f_i^k)_{i=1}^{N_k}$, where k represents the discretization level, the decimation operator D_k^{k-1} computes $f^{k-1} = (f_i^{k-1})_{i=1}^{N_{k-1}}$, at the next coarser discretization level ($N_{k-1} < N_k$). The prediction operator made an approximation $\tilde{f}^k = (\tilde{f}_i^k)_{i=1}^{N_k}$ to $f^k = (f_i^k)_{i=1}^{N_k}$ from $f^{k-1} = (f_i^{k-1})_{i=1}^{N_{k-1}}$.

The decimation operator is always assumed to be linear. In contrast, the prediction operator need not be linear, should at least satisfy the consistency requirement $D_k^{k-1} \cdot P_{k-1}^k = I_{N_{k-1}}$, where $I_{N_{k-1}}$ denotes the identity operator in $\mathrm{R}^{N_{k-1}}$. If a nonlinear operator is considered as prediction we will obtain a nonlinear multiresolution. From the consistency property, it follows that the null space of D_k^{k-1} has dimension $N_k - N_{k-1}$, since the image of D_k^{k-1} is the full $\mathrm{R}^{N_{k-1}}$. Then we can decompose the prediction error according to

$$f^k - \tilde{f}^k = \sum_{i=1}^{N_k - N_{k-1}} d_i^{k-1} e_i^{k-1} \tag{3}$$

where $(e_i^{k-1})_{i=1}^{N_k - N_{k-1}}$ is a basis of W_k (space of the details defined as the null space of the prediction operator).

By iteration of this process from $k = L$ to $k = 1$, we obtain a multiscale decomposition of f^L into $(f^0, d^0, d^1, \ldots, d^{L-1})$.

Let G_k be the operator which computes the coordinates of the prediction error in a basis of $\mathcal{N}(D_k^{k-1})$, E_k such that $e^k = E_k G_k e^k$. Then the direct and inverse transforms of the multiresolution process take the form

$v^L \quad \to \quad Mv^L \quad$ (Encoding)

$$\begin{cases} \text{Do} \quad k = L, \ldots, 1 \\ v^{k-1} = D_k^{k-1} v^k \\ d^k = G_k(v^k - P_{k-1}^k v^{k-1}) \end{cases} \tag{4}$$

$$Mv^L = \{v^0, d^1, \ldots, d^L\}$$

$Mv^L \quad \to \quad M^{-1}Mv^L \quad$ (Decoding)

$$\begin{cases} \text{Do} \quad k = 1, \ldots, L \\ v^k = P_{k-1}^k v^{k-1} + E_k d^k \end{cases} \tag{5}$$

Proceedings of the 5th WSEAS Int. Conf. on Signal Processing, Robotics and Automation, Madrid, Spain, February 15-17, 2006 (pp16-21)

On the other hand, in practice the prediction operator (the decimation operator also) is constructed by using two fundamental tools: discretization and reconstruction. The discretization $\mathcal{D}_k$ is a linear operator that connects a functional space $\mathcal{F}$ with the space V^k and yields discrete information at the resolution level k specified by a grid X^k. The reconstruction operator $\mathcal{R}_k$ goes from V^k to $\mathcal{F}$. A basic consistency requirement is that

$$\mathcal{D}_k \mathcal{R}_k f^k = f^k \tag{6}$$

Given sequences of discretization and reconstruction operators satisfying (6), it is then possible to define the decimation and prediction operators according to

$$D_{k-1}^k = \mathcal{D}_{k-1} \mathcal{R}_k. \tag{7}$$

$$P_k^{k-1} = \mathcal{D}_k \mathcal{R}_{k-1}. \tag{8}$$

Remark 1 *If $\{\mathcal{D}_k\}$ is a nested sequence, that is, if for all k and all $f \in \mathcal{F}$*

$$\mathcal{D}_k f = 0 \rightarrow \mathcal{D}_{k-1} f = 0$$

the dependence of D_{k-1}^k on the reconstruction $\mathcal{R}_k$ is totally fictitious.

This description of the prediction operator opens up a great number of possibilities in designing subdivision schemes. The reconstruction process is the key step, and nonlinear reconstructions operators will lead to nonlinear subdivision schemes. The main concern in this design should be the "quality" of the prediction P_{k-1}^k. The quality of the prediction operator can be measured by the set of functions for which the reconstruction is *exact*. Thus nonlinear techniques appear as good strategies.

We finalize the section with a brief description of the reconstruction process associated to the point-values and cell-average frameworks in $[0,1]$ (the details can be seen in [13]).

2.1 Point-value schemes in 1D

Let us consider a set of nested grids:

$$X^k = \{x_i^k\}_{i=0}^{J_k}, \quad x_i^k = ih_k, \quad h_k = 2^{-k}/J_0,$$

where $J_k = 2^k J_0$, J_0 a fixed integer. Consider the point-value discretization

$$D_k : C([0,1]) \rightarrow V^k, \quad f_i^k = (D_k f)_i := f(x_i^k),$$

where V^k is the space of sequences of dimension $N_k = J_k + 1$. A reconstruction procedure for this discretization operator is given by an operator R_k such that

$$R_k : V^k \rightarrow C([0,1]); \quad D_k R_k f^k = f^k \tag{9}$$

which means that

$$(R_k f^k)(x_i^k) = f_i^k = f(x_i^k), \tag{10}$$

therefore, $(R_k f^k)(x)$ should be a continuous function that interpolates the data f^k on X^k.

Thus if we denote by $I_k(x; f^k)$ such an interpolatory reconstruction of the data f^k. The predictor operator can be computed as follows:

$$(P_{k-1}^k f^{k-1})_i = I_{k-1}(x_i^k; f^{k-1}) \tag{11}$$

A nonlinear interpolatory technique will lead to a nonlinear multiresolution scheme [11]-[13]-[1].

3 Multiresolution packets

In this section we shall introduce the general "Multiresolution packet". The same as the library of wavelet packet bases it is naturally organized as subsets of binary tree. This segmentation of signals into those dyadic intervals is better adapted to the frequency content. The idea is to obtain the best decomposition of all the possible ones. We now define a cost function on sequence and search for its minimum over all representation in a library. For a given vector, their minima are the most efficient representation.

Definition 1 *A map $\mathcal{L}$ from sequences $\{x_j\}$ to R is called an additive information cost function if $\mathcal{L}(0) = 0$ and $\mathcal{L}(\{x_j\}) = \sum_j \mathcal{L}(x_j)$.*

Some useful examples of information cost include: a)Number above a threshold, set an arbitrary threshold ϵ and count the elements in the sequence x whose absolute value exceeds ϵ. b) Concentration in l^p norm ($p < 2$), $\mathcal{L}(x) = ||x||_p$. c) Entropy, $\mathcal{L}(x) = -\sum_j p_j log p_j$ where $p_j = \frac{|x_j|^2}{||x||^2}$ and we set $p\ log p = 0$ if $p = 0$. d) Logarithm of energy, $\mathcal{L}(x) = \sum_j log|x_j|^2$. For more details see [9]. Here we use the first possibility.

As the library is a tree, then we can find the best representation by induction on the number of scales. Denote by s_j^k the representation of vectors corresponding to the scale k, $j = 0, 1, 2, \ldots, 2^{L-k} - 1$, and by $\mathcal{B}_j^k$ the best representation for x.

Whenever a parent node is of lower information cost than the children, we mark the parent. In the final representation we have all the information, that is, the value of the details and the marks.

In practice, we start with a vector of data $s_0^L = f^k$, corresponding to any discretization of a certain function. We compute a step of the multiresolution algorithm, that is, $s_0^{L-1} = f^{k-1}$ and the details $s_1^{L-1} = d^k$. If the addition of the cost of these two new vectors are higher than it comes from s_0^L we do not consider the decomposition. On the other hand, if the cost is minor then we carry out the decomposition. If the last case has been produced then we would repeat the process for these two new vectors (s_0^{L-1} and s_1^{L-1}) independently. Anyhow, the decomposition is finished when one has arrived to the worst resolution level prescribed by the user. In the framework of Harten, the one to one correspondence between two discretization levels, when $\mathcal{L}(\mathcal{B}_j^k) > \mathcal{L}(s_{2j}^{k-1}) + \mathcal{L}(s_{2j+1}^{k-1})$, is given by

$$s_j^{k-1} = \begin{cases} D_k^{k-1}(s_{\frac{j}{2}}^k) & j \text{ even} \\ G_k Q_k(s_{\frac{j-1}{2}}^k) & \text{otherwise} \end{cases} \tag{12}$$

$$s_j^k = P_{k-1}^k(s_{2j}^{k-1}) + E_k(s_{2j+1}^{k-1}) \tag{13}$$

4 The contraction property and the stability result

We focus on the subdivision scheme S associated to the prediction that writes

$$f^{k-1} \to S(f^{k-1})f^{k-1} = \mathcal{D}_k\mathcal{R}_{k-1}f^{k-1},$$

with

$$\begin{cases} (\mathcal{D}_k\mathcal{R}_{k-1}f^{k-1})_{2j+1} = P_{k-1}^k(x_{j+\frac{1}{2}}^k), \\ (\mathcal{D}_k\mathcal{R}_{k-1}f^{k-1})_{2j} = f_j^{k-1}. \end{cases} \tag{14}$$

We assume the following contractivity property:

Definition 2 *We say that the subdivision scheme S has a contractivity property if for all k*

$$\begin{aligned} ||f^k - g^k||_{l_\infty(Z)} &\leq ||f^{k-1} - g^{k-1}||_{l_\infty(Z)} \\ &+ C||D(f^{k-1} - g^{k-1})||_{l_\infty(Z)}. \end{aligned}$$

and

$$||D(f^k - g^k)||_{l_\infty(Z)} \leq \rho||D(f^{k-1} - g^{k-1})||_{l_\infty(Z)}.$$

where C is a positive constant, $\rho \in (0,1)$ and D is a linear operator.

We have the following theorem related to the stability of the reconstruction.

Theorem 1 *Given $\{s_0^0, s_1^0, \ldots, s_{2^L-1}^0\}$ and $\{\bar{s}_0^0, \bar{s}_1^0, \ldots, \bar{s}_{2^L-1}^0\}$ two decompositions, corresponding to $s_0^L, \bar{s}_0^L \in l_\infty(Z)$ respectively, of a multiresolution-packets scheme that verifies a contractivity property, then we have*

$$||s_0^L - \bar{s}_0^L||_{l_\infty(Z)} \leq (1 + \frac{C\tilde{C}}{1-\rho}) \sum_{k=1}^{L} ||d^k - \bar{d}^k||.$$

where $\tilde{C}$ is a bound of $||D||_{l_\infty(Z)}$.

References

[1] S.Amat, F.Aràndiga, A.Cohen, R.Donat, G.García and M.von Oehsen. *Data compresion with ENO schemes: A case study.* Applied and Computational Harmonic Analysis, **11**, 273-288, (2001).

[2] S.Amat, K.Dadourian and J.Liandrat. *Nonlinear Subdivision Schemes and Associated Multiresolution Transforms.* In preparation.

[3] S.Amat and J.Liandrat. *On the stability of PPH nonlinear multiresolution.* Applied and Computational Harmonic Analysis. **18** (2), 198-206, (2005).

[4] F.Aràndiga and R.Donat. *Nonlinear Multi-scale Decompositions: The approach of A.Harten.* Numerical Algorithms, **23**, pp. 175-216, (2000).

[5] G. Beylkin. *Wavelets, Multiresolution Analysis and Fast Numerical Algorithms.* INRIA lectures, manuscript, (1991).

[6] G. Beylkin, R. Coifman and V. Rokhlin. *Fast Wavelet Transform and Numerical Algorithms I.* Comm. Pure Appl. Math., **XLIV**, pp. 141-183, (1991).

[7] R.R.Coifman, Y.Meyer, Y.Quake, M.V.Wickerhauser. *Signal processing and compression with wavelet packets*, 'Progress in Wavelet Analysis and Applications ', Y.Meyer and S.Roquesed., Editions Frontires (1993).

[8] R.R.Coifman, Y.Meyer, M.V.Wickerhauser. *Size properties of the wavelet packets.* 'Wavelets and their applications', Ruskai et al. (ed.), Jones and Barlet, pp. 453-470, (1992).

[9] R.R.Coifman, M.V.Wickerhauser. *Entropy-based methods for best basis selection.* IEEE Trans. on Inf. Theory, **28-2**, pp. 719-746, (1992).

[10] I.Daubechies. *Ten Lectures on Wavelets.* CBMS-NSF Regional Conference Series in Applied Mathematics (1992).

[11] A. Harten. *Discrete Multiresolution Analysis and Generalized Wavelets.* J. Appl. Numer. Math., **12**, pp. 153-192, (1993).

[12] A. Harten. *Multiresolution Representation of Data.* UCLA CAM Report 93-13 (1993).

[13] A. Harten. *Multiresolution Representation of Data II: General Framework.* SIAM J. Numer. Anal. **33** #3 pp. 1205-1256, (1996).

[14] P.Joly, Y.Maday, V.Perrier. *Towards a Method for Solving Partial Differential Equations by Using Wavelet Packet Bases.* Comput. Methods in Appl. Mech. and Engrg. **116**, pp. 301-307, (1994).

[15] X.D.Liu, S.Osher and T.Chan. *Weighted essentially non-oscillatory schemes.* Journal Computational Physics, **115**, pp. 200-212, 1994.

[16] M.V.Wickerhauser. *Picture compression by best-basis sub-band coding.* Prepint, Yale University (New Haven, Connecticut), (1990).