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Detection, measurement and classification of discontinuities

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Abstract: This paper is devoted to the detection, measurement and classification of discontinuities. A nonlinear scheme is proposed. Some numerical experiments are presented.

Key Words: Edges' detection, noise, nonlinear schemes.

1 Introduction

The problem of detection, classification and measurement of discontinuities appears in many applications in science and technology. Some of these processes produce piecewise smooth data, that is functions with a small number of discontinuities compared to the number of sampled data. Assume that the input function is corrupted by an additive random noise $\hat{f} = f + n$, we would like to find the discontinuities of the signal f. The noise disturbs the data thus the problem is complex. It is difficult to distinguish the true discontinuities from the function and the false discontinuities from the noise. The noise added to the signal appears as small oscillatory deviations from the curve. A non-linear detector algorithm is presented. The main advantage of our algorithm is that we can consider noise larger than the classic methods.

A function has a discontinuity of degree k at a point, if the k th-order left and right derivatives at that point are different. Discontinuities are classified by their degrees and measured by their sizes, that is, the difference of the derivatives.

In [3], divided differences were studied to obtain the possible discontinuities. Our detection algorithm is based on this study which uses the subcell resolution technique introduced by Harten [1]. However, when we consider signals corrupted by noise, we have to modify the detecting mechanisms, since the algorithm should be adapted to the introduced noise. Our new algorithm detects true singularities only and not singularities introduced by the noise. In the examples we will see that it is possible to consider noise of very large size.

Now we will recall the subcell resolution technique.

2 The Subcell Resolution Technique.

Let us assume that $H(x)$ is a continuous function with a corner at $x_d \in (x_{j-1}, x_j)$. Then, the ENO polynomials [1] satisfy

$$H(x) = q_{j-1}(x) + O(h^r), \quad x \in [x_{j-2}, x_{j-1}] \tag{1}$$

$$H(x) = q_{j+1}(x) + O(h^r), \quad x \in [x_j, x_{j+1}] \tag{2}$$

The location of the corner, x_d, can be recovered using the following function:

$$G_j(x) = q_{j+1}(x) - q_{j-1}(x) \tag{3}$$

Using Taylor expansion in regions of smoothness, it is not hard to prove that

$$G_j(x_{j-1}) \times G_j(x_j) = a(a-1)[H']^2_{x_d} h^2 + O(h^3)$$

where $x_d = x_j - ah$, $0 < a < 1$ and $[H']_{x_d}$ denotes the jump of the derivative at x_d.

Therefore, if h is sufficiently small, there is a root of G_j in (x_{j-1}, x_j) be such that $G_j(\theta_j) = 0$. In general, it can be proven [1] that

$$|\theta_j - x_d| = O(h^r)$$

Remark 2.1 *If the function is a piecewise polynomial with a corner in x_d then $x_d = \theta_j$.*

Working with cell-average and hat-average, via first and second primitive (see [2] and [3]), we can detect jumps and delta singularities. With these multiresolutions we can detect "weaker" singularities also, that is, with the cell-average we can detect corners and with the hat-average we can detect corners and jumps. In these cases it becomes very important to isolate cells that are suspected of harboring a singularity.

On the other hand, we know that when $H(x)$ has a discontinuity in its $m+1$st derivative at $x_d \in (x_{j-1}, x_j)$, it can be approximated (for sufficiently small h) by the unique root of $G_j^{(m)}(z) = q_{j+1}^{(m)}(z) - q_{j-1}^{(m)}(z) = 0$. Thus, if (x_{j-1}, x_j) is suspected of containing a singularity (stencil selection, see [3]), we check whether

$$G_j^{(m)}(x_{j-1}) \cdot G_j^{(m)}(x_j) < 0. \tag{4}$$

If this is the case, we conclude that there is a root of $G_j^{(m)}(z)$ in (x_{j-1}, x_j).

A careful analysis of the functions $G_j^{(m)}(x)$ for $m = 0, 1, 2$ can help to determine whether or not a singularity lies at a *suspicious* grid point.

Since we are interesting in jumps and corners we will consider the cell-average framework.

2.1 Full detection mechanism

As we said before, we will work with signals perturbed with noise for which we will assume some conditions. For our algorithm we will need to know some bound ϵ of the introduced noise. The knowing of noise bounds is not a big restriction. In the classic detectors, it is supposed that the noise is modelled by a gaussian of which we know the mean μ and the variance σ. With this information we can find bounds since $Pr(f \in [\mu - 2\sigma, \mu + 2\sigma]) = 0.95$.

When we don't have any information of the noise (for example picture from an airplane with a lot of fog), we can generate a decreasing succession of parameters ϵ_k in order to obtain the possible discontinuities. We keep the discontinuities obtained in a such ϵ_{k_0} if the next parameter ϵ_{k_0+1} knowledge the detection mechanism getting a number limitless of discontinuities.

	$x_d \in (x_{j-1}, x_j)$	
z	$G_j(z)$	$G'_j(z)$
x_{j-1}	$(a-1)h[H']_{x_d}$	$[H']_{x_d}$
x_j	$ah[H']_{x_d}$	$[H']_{x_d}$
x_{j+1}	$(a+1)h[H']_{x_d}$	$[H']_{x_d}$

Table 1: Jump in $f(x)$ ($[H']_{x_d} \neq 0$).

	$x_d = x_j$	
z	$G_j(z)$	$G'_j(z)$
x_{j-1}	$-h[H']_{x_d}$	$[H']_{x_d}$
x_j	$O(h^{p+2})$	$[H']_{x_d}$
x_{j+1}	$h[H']_{x_d}$	$[H']_{x_d}$

Table 2: Jump in $f(x)$ ($[H']_{x_d} \neq 0$).

Tables 1, 2, 3 and 4 (which are constructed via Taylor expansions) reflect the behavior of the functions $G^{(m)}$ near the different types of singularities (see [3] for more details). When we are considering signals perturbed with

	$x_d \in (x_{j-1}, x_j)$	
z	$G'_j(z)$	$G''_j(z)$
x_{j-1}	$(a-1)h[H'']_{x_d}$	$[H'']_{x_d}$
x_j	$ah[H'']_{x_d}$	$[H'']_{x_d}$
x_{j+1}	$(a+1)h[H'']_{x_d}$	$[H'']_{x_d}$

Table 3: Corner in $f(x)$ ($[H'']_{x_d} \neq 0$).

	$x_d = x_j$	
z	$G'_j(z)$	$G''_j(z)$
x_{j-1}	$-h[H'']_{x_d}$	$[H'']_{x_d}$
x_j	$O(h^{p+1})$	$[H'']_{x_d}$
x_{j+1}	$h[H'']_{x_d}$	$[H'']_{x_d}$

Table 4: Corner in $f(x)$ ($[H'']_{x_d} \neq 0$).

noise, the problem is more difficult. The idea is to study how this noise affects the divided differences. Notice you for example that if $||f_i - \hat{f}_i|| < \epsilon$ then for the divided differences of order 4 we have $||H[i;4] - \hat{H}[i;4]|| < \frac{8}{6}\frac{\epsilon}{h^3}$.

Our strategy to detect singularities is based on modifications of tables 1, 2, 3 and 4 regarding the presence of noise.

REMARK 2.2 *We only need a vector of data not the complete function.*

2.2 Numerical experiments and Conclusions

In this section we will introduce some plots of modified signals and pictures by random noise. Our algorithm detects the real singularities with a great noise. Our scheme detects only true discontinuities. When the noise used is too big (for which the true discontinuities and those taken place by the noise have the same characteristics) our algorithm doesn't detect anything. Nevertheless, we can see in our examples that we can consider very large noises, more than the classical detectors.

In our experiments we use the idea of decreasing sequence to detect the discontinuities. After we check the measure of discontinuity and we decide the true singularities if the size is big enough with respect the noise. As we said before, we will consider the cell-average framework.

We start in 1-D with a jump and a corner. In figure 1, we plot the signals and in figure 2, the perturbation. We use 64 points, and the noise is lees than 0.4 and 0.01 respectively. In table 5 we can see the good resolution of the detector.

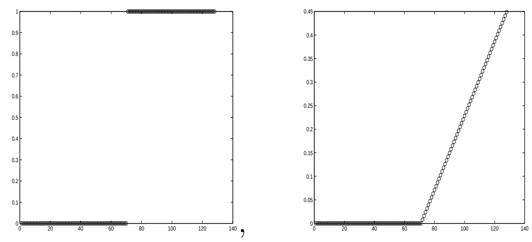

Figure 1: left jump, right corner

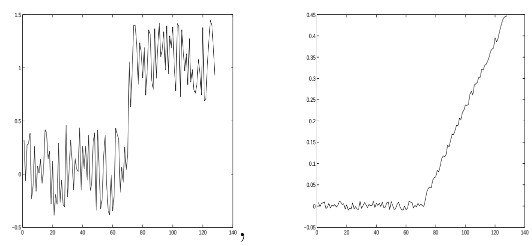

Figure 2: left jump with noise, right corner with noise

	fig.2 left	fig.2 right	fig.3 left	fig.3 right
cell	$(35,36)$	$(35,36)$	$(35,36)$	$(35,36)$
location	0.547	0.547	0.499	0.510
size	1.	1.	1.	1.1
type	jump	corner	jump	corner

Table 5: $n = 64$, cell-average

Next, we apply our detector to images in 2-D. In figure 3, we consider a geometric picture without noise, and in figure 4 a perturbation of 3 with a noise lees than 10 is considered. We detect all the jumps, in table 6 we display their sizes (the real sizes are 50 in all the cases).

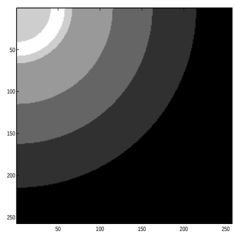

Figure 3: noise=0

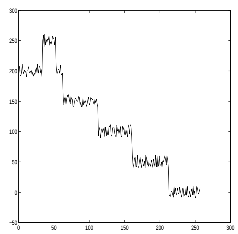

Figure 4: section with noise

first jump	second jump	third jump
65	60	50
fourth jump	fifth jump	sixth jump
55	65	65

Table 6: $n = 256$, cell-average, noise=10

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