Proceedings of the 5th WSEAS Int. Conf. on Artificial Intelligence, Knowledge Engineering and Data Bases, Madrid, Spain, February 15-17, 2006 (pp106-116)

Splitting Algorithm for Solving an Integral Differential Equation

NIKOS MASTORAKIS
Head of the Department of Computer Science
Military Inst. of University Education / Hellenic Naval Academy
Terma Hatzikyriakou 18539,
Piraeus, GREECE
http://www.wseas.org/mastorakis

OLGA MARTIN
Department of Mathematics
University "Politehnica" of Bucharest
Splaiul Independentei 313
Bucharest, ROMANIA

Abstract. The initial-boundary value problem for a non-stationary transport equation is considered. This is rewritten as a Cauchy problem: $d\varphi/dt + A\varphi = F$, $\varphi|_{t=0} = \varphi_0$. The unknown φ represents a suitable subset of a Hilbert space, whose elements are pairs of real-valued functions depending on three variables: a space variable $z\in[0, H]$, an angle variable ν, with $\mu = \cos \nu \in [0, 1]$ and a time variable $t\in[0, T]$. A difference scheme is given in order to approximate the space derivatives appearing in A. Then, the operator A is decomposed as $A = A_1 + A_2$ and another difference scheme is given to approximate the time derivatives. Finally, the numerical integration with respect to μ is carried out. One obtains an algorithm, which approximates the exact solution with an accuracy of second order in time step τ and in space step h. Several numerical examples are included.

Key words: transport equation, difference scheme, Krank-Nicholson scheme, bicycle splitting-up method.

1 Introduction

The main problem in the nuclear physics is to find the neutron distribution in the reactor, hence its density, φ. This is a scalar function, which is studied in a plan-parallel geometry and depends on the next variables: the position of the neutron on the Oz – axis, the neutron speed, which makes an angle ν with Oz and the time, t. The density is the solution of an integral differential equation, named the non-stationary neutron transport equation.

Many authors paid attention to this problem, [2],[5],[6],[10],[14], but their papers are theoretical studies. In this paper we present an algorithm inspired by a splitting-up method, [5], applied to a non-stationary transport equation. In the general case, this method is hard to use, but for any symmetry of the source function, it leads to an algorithm flexible. We prove that the operators of the problem are positive. Also, we determine an aprioristic estimation of the solution. The study of the approximation of the solution with respect to time step, τ, shows that it is of the τ^2 order. The numerical examples prove that the errors, which correspond of the approximate solutions, are minimum.

2 Problem formulation

Let us consider a transport equation in a plan – parallel geometry:

$$\frac{1}{v_c}\cdot\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial z} + \sigma\cdot\varphi = \frac{\sigma_s}{2}\int_{-1}^{1}\varphi d\mu + f(z,\mu,t) \quad (1)$$

with the following boundary conditions:

$$\begin{aligned} &\varphi = 0 \ \text{ if } z = 0,\ \mu > 0 \\ &\varphi = 0 \ \text{ if } z = H,\ \mu < 0 \end{aligned} \quad (2)$$

and the initial condition:

$$\varphi = \varphi^0 \text{ if } t = 0. \tag{3}$$

The unknown φ of the problem (1) – (3) is the density of the particles. These move with the speed v_c, which makes an angle ν with the real axis Oz at moment t and $\mu = \cos\nu$. The right-hand side of (1), f, is the radioactive source, the functions σ, σ_s are continuous in the interval $[0, H]$ and satisfy the conditions:

$$0 < \sigma_0 \le \sigma \le \sigma_1 < \infty; \quad 0 \le \sigma_s \le \sigma_s' < \infty;$$
$$0 < \sigma_{c_0} \le \sigma_c = \sigma - \sigma_s \tag{4}$$

Further on, we consider for simplicity, $v_c = 1$. Using the notations:

$$\varphi^+ = \varphi(z, \mu, t); \; \varphi^- = \varphi(z, -\mu, t) \tag{5}$$

where $\mu > 0$, the equation (1) can be rewritten in the form:

$$\frac{\partial\varphi^+}{\partial t} + \mu\frac{\partial\varphi^+}{\partial z} + \sigma\cdot\varphi^+ = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu + f^+$$
$$\frac{\partial\varphi^-}{\partial t} - \mu\frac{\partial\varphi^-}{\partial z} + \sigma\cdot\varphi^- = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu + f^- \tag{6}$$

Substituting: $\mu' = -\mu > 0$, we get:

$$\int_{-1}^0 \varphi(z,\mu,t)d\mu = -\int_1^0 \varphi(z,-\mu',t)d\mu' =$$
$$= \int_0^1 \varphi(z,-\mu',t)d\mu' = \int_0^1 \varphi^- d\mu$$

The boundary value problem becomes:

$$\varphi^+(0,\mu,t) = 0; \quad \varphi^-(H,\mu,t) = 0,$$
$$\forall\mu \in [0,1], \; \forall t \in [0,T] \tag{7}$$

Adding and subtracting the equations (6) and introducing the notations:

$$u = \frac{1}{2}(\varphi^+ + \varphi^-) \qquad g = \frac{1}{2}(f^+ + f^-)$$
$$v = \frac{1}{2}(\varphi^+ - \varphi^-) \qquad r = \frac{1}{2}(f^+ + f^-) \tag{8}$$

we obtain the following system:

$$\frac{\partial u}{\partial t} + \mu\cdot\frac{\partial v}{\partial z} + \sigma\cdot u = \sigma_s\int_0^1 u\,d\mu' + g$$
$$\frac{\partial v}{\partial t} + \mu\cdot\frac{\partial u}{\partial z} + \sigma\cdot v = r. \tag{9}$$

The boundary - initial conditions are:

$$u + v = 0 \text{ for } z = 0$$
$$u - v = 0 \text{ for } z = H \tag{10}$$

and respectively:

$$u = u^0, \; v = v^0 \text{ for } t = 0. \tag{11}$$

Now we rewrite the problem (9)-(11) in a operator form. For this purpose, we introduce the vector functions having two scalar components:

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \quad F = \begin{pmatrix} g \\ r \end{pmatrix}. \tag{12}$$

and the operator

$$A = \begin{pmatrix} \sigma - \sigma_s\int_0^1 d\mu' & \mu\frac{\partial}{\partial z} \\ \mu\frac{\partial}{\partial z} & \sigma \end{pmatrix} \tag{13}$$

Let us define in the measurable set $D = [0, H]\times[0,1]$, a Hilbert space $L_2(D)$ with the scalar product for every fixed t:

$$(\alpha(t),\beta(t)) = \sum_{i=1}^2\int_0^1 d\mu\int_0^H \alpha^i(z,\mu,t)\beta^i(z,\mu,t)dz \tag{14}$$

where α^i, β^i are the components of the vectors functions α, β. Then, we shall isolate in $L_2(D)$ a set Φ in the following manner:

$$\Phi = \{w \,|\, w \in L_2(D), (Aw, w) < \infty\}$$

Finally, we consider $\Phi_0 \subset \Phi$, the subset of functions w with the next properties: are continuous, have $\partial w/\partial z$ continuous on D and the components u, v verify the conditions (10).

Let us define the operator

$$L = \frac{\partial}{\partial t} + A \tag{15}$$

with the domain, $D(L) = \Phi_0$.
Consequently, the problem (9) – (11) becomes:

$$\frac{\partial w}{\partial t} + Aw = F \tag{16}$$

$$(z,\mu,t) \in D \times [0,T], \quad D = [0,H] \times [0,1]$$

$$w(z,\mu,0) = w^0, \quad \forall (z,\mu) \in D \tag{17}$$

$$F \in L_2(D \times [0,T]), \quad w^0 \in \Phi,\ w \in \Phi_0.$$

We prove that A positive operator, namely, $(Aw,w) > 0$ for each $w \neq 0$. We have

$$(Aw,w) =$$
$$\int_0^1 d\mu \int_0^H \left[\sigma u^2 - \sigma_s u \int_0^1 u d\mu' + \mu u \frac{\partial v}{\partial z} + \mu v \frac{\partial u}{\partial z} + \sigma v^2 \right] dz \tag{18}$$

Using the Hölder inequality, we obtain

$$\left(\int_0^1 1 \cdot u d\mu \right)^2 \le \left(\int_0^1 1^2 d\mu \right)\left(\int_0^1 u^2 d\mu \right) = \int_0^1 u^2 d\mu$$

Finally, for $\sigma_s \le \sigma$ we get

$$(Aw,w) \ge \sigma_c \int_0^H \left(\left(\int_0^1 u d\mu \right)^2 - \left(\int_0^1 u d\mu \right)^2 \right) dz +$$
$$\int_0^1 d\mu \int_0^H \left(\sigma v^2 + \mu \frac{\partial}{\partial z}(uv) \right) dz \ge \tag{19}$$
$$\ge \frac{1}{4} \int_0^1 \mu [(\varphi^+)^2 - (\varphi^-)^2]_0^H d\mu =$$
$$= \frac{1}{4} \int_0^1 \mu ((\varphi^+)^2 + (\varphi^-)^2) d\mu > 0$$

according with (7). If the operator L is positive, then the equation:

$$Lw = F$$

has only one solution. Indeed, let $w_1 \neq w$ be an element such that

$$Lw_1 = F.$$

Hence, $L(w - w_1) = 0 \Rightarrow L\overline{w} = 0 \Rightarrow (L\overline{w}, \overline{w}) = 0$. The operator L is positive, such that $\overline{w} = 0 \Rightarrow w = w_1$.

In order to get a solution of the problem (16), we go through three stages.

First, a difference scheme is given in order to approximate the space derivatives which appearing in A. We consider on z – axis two points systems:

- a principal system, $\{z_k\}_k$, $k \in \{0,1,...,N\}$ with $z_0 = 0$ and $z_N = H$;
- a secondary system, $\{z_{k+1/2}\}_k$, $k \in \{0,1,...,N\text{-}1\}$, which verifies the inequality: $z_{k-1/2} < z_k < z_{k+1/2}$.

Integrating the first equation (9) on the intervals: $(z_0, z_{1/2})$, $(z_{k-1/2}, z_{k+1/2})$, $k \in \{1, 2,\dots, N\text{-}1\}$, $(z_{N-1/2}, z_N)$ and the second equation on (z_{k-1}, z_k), $k \in \{1, 2,\dots, N\text{-}1\}$, the system can be written in the form:

$$\frac{\partial}{\partial t}\int_{z_0}^{z_{1/2}} u\,dz + \mu \int_{z_0}^{z_{1/2}} \frac{\partial v}{\partial z} dz + \int_{z_0}^{z_{1/2}} \sigma u\,dz =$$
$$= \int_{z_0}^{z_{1/2}} \sigma_s\,dz \int_0^1 u\,d\mu' + \int_{z_0}^{z_{1/2}} g\,dz \tag{20}$$
$$\frac{\partial}{\partial t}\int_{z_0}^{z_1} v\,dz + \mu \int_{z_0}^{z_1} \frac{\partial u}{\partial z} dz + \int_{z_0}^{z_1} \sigma v\,dz = \int_{z_0}^{z_1} r\,dz$$

……………………………………………………

$$\frac{\partial}{\partial t}\int_{z_{k-1/2}}^{z_{k+1/2}} u\,dz + \mu \int_{z_{k-1/2}}^{z_{k+1/2}} \frac{\partial v}{\partial z} dz + \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma u\,dz =$$
$$= \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma_s\,dz \int_0^1 u\,d\mu' + \int_{z_{k-1/2}}^{z_{k+1/2}} g\,dz$$
$$\frac{\partial}{\partial t}\int_{z_k}^{z_{k+1}} v\,dz + \mu \int_{z_k}^{z_{k+1}} \frac{\partial u}{\partial z} dz + \int_{z_k}^{z_{k+1}} \sigma v\,dz = \int_{z_k}^{z_{k+1}} r\,dz \tag{21}$$

……………………………………………………

$$\frac{\partial}{\partial t}\int_{z_{N-1/2}}^{z_N} u\,dz + \mu \int_{z_{N-1/2}}^{z_N} \frac{\partial v}{\partial z} dz + \int_{z_{N-1/2}}^{z_N} \sigma u\,dz =$$
$$= \int_{z_{N-1/2}}^{z_N} \sigma_s\,dz \int_0^1 u\,d\mu' + \int_{z_{N-1/2}}^{z_N} g\,dz$$
$$\frac{\partial}{\partial t}\int_{z_{N-1}}^{z_N} v\,dz + \mu \int_{z_{N-1}}^{z_N} \frac{\partial u}{\partial z} dz + \int_{z_{N-1}}^{z_N} \sigma v\,dz = \int_{z_{N-1}}^{z_N} r\,dz$$

With the following notations:

$$\begin{aligned} &\Delta z_0 = z_{1/2} - z_0, \quad \Delta z_k = z_{k+1/2} - z_{k-1/2}, \\ &k = 1,2,.., N-1 \\ &\Delta z_N = z_N - z_{N-1/2}, \quad \Delta z_{k-1/2} = z_k - z_{k-1}, \\ &k = 1,2,..., N \end{aligned} \tag{22}$$

we define the mean values:

$$\sigma_k = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma \, dz\,, \qquad \sigma_{k+1/2} = \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma \, dz,$$

$$\sigma_{s_k} = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma_s \, dz \qquad (23)$$

$$g_k = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} g \, dz, \qquad r_{k+1/2} = \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma \, dz$$

Let $h = \max_{0 \le k \le N-1} (\Delta z_0, \Delta z_N, \Delta z_k, \Delta z_{k+1/2})$. Dividing the first equation (20) by Δz_0 and the second by $\Delta z_{1/2}$ we obtain:

$$\frac{\partial}{\partial t}\left[\frac{1}{\Delta z_0}\int_{z_0}^{z_{1/2}} u \, dz\right] + \mu \frac{1}{\Delta z_0}[v]_{z_0}^{z_{1/2}} +$$

$$+ \frac{1}{\Delta z_0}\int_{z_0}^{z_{1/2}} \sigma(z)\, u(z,\mu,t)\, dz =$$

$$= \int_0^1 \left[\frac{1}{\Delta z_0}\int_{z_0}^{z_{1/2}} \sigma_s u \, dz\right] d\mu' + \frac{1}{\Delta z_0}\int_{z_0}^{z_{1/2}} g \, dz \qquad (24)$$

$$\frac{\partial}{\partial t}\left[\frac{1}{\Delta z_{1/2}}\int_{z_0}^{z_1} v \, dz\right] + \mu \frac{1}{\Delta z_{1/2}}[u]_{z_0}^{z_1} +$$

$$+ \frac{1}{\Delta z_{1/2}}\int_{z_0}^{z_1} \sigma \, v \, dz = \frac{1}{\Delta z_{1/2}}\int_{z_0}^{z_1} r \, dz$$

In accordance with the boundary conditions:

$$v_0 = v\big|_{z=0} = \frac{1}{2}\left(\varphi^+ - \varphi^-\right)\Big|_{z=0} = -\frac{1}{2}\varphi^- \qquad (25)$$

$$u_0 = u\big|_{z=0} = \frac{1}{2}\left(\varphi^+ + \varphi^-\right)\Big|_{z=0} = \frac{1}{2}\varphi^- \qquad (26)$$

we have: $v_0 = -\, u_0$ and the relation (24) can be rewritten in the form:

$$\frac{\partial u_0}{\partial t} + \mu \frac{v_{1/2} + u_0}{\Delta z_0} + \sigma_0 u_0 = \sigma_{s_0}\int_0^1 u_0 d\mu' + g_0,$$

$$g_0 = \frac{1}{\Delta z_0}\int_{z_0}^{z_{1/2}} g dz \qquad (27)$$

$$\frac{\partial v_{1/2}}{\partial t} + \mu \frac{u_1 - u_0}{\Delta z_{1/2}} + \sigma_{1/2} v_{1/2} = r_{1/2}$$

where the functions u, v are replaced by their values in the points: $z = 0$, $z = 1/2$, $z = 1$ and by mean values. Similarly, we get

$$\frac{\partial u_k}{\partial t} + \mu \frac{v_{k+1/2} - v_{k-1/2}}{\Delta z_k} + \sigma_k u_k =$$

$$= \sigma_{s_k}\int_0^1 u_k d\mu' + g_k$$

$$(28)$$

$$\frac{\partial v_{k+1/2}}{\partial t} + \mu \frac{u_{k+1} - u_k}{\Delta z_{k+1/2}} + \sigma_{k+1/2} v_{k+1/2} = r_{k+1/2},$$

$$k = 1,2,\ldots, N-1$$

$$\frac{\partial v_{N-1/2}}{\partial t} + \mu \frac{u_N - u_{N-1}}{\Delta z_{N-1/2}} + \sigma_{N-1/2} v_{N-1/2} = r_{N-1/2}$$

$$(29)$$

$$\frac{\partial u_N}{\partial t} + \mu \frac{u_N - v_{N-1/2}}{\Delta z_N} + \sigma_N u_N = \sigma_{s_N}\int_0^1 u_N d\mu' + g_N$$

$$g_N = \frac{1}{\Delta z_N}\int_{z_{N-1/2}}^{z_N} g dz$$

where $u_N = v_N$.

Let us consider $M(0, 2N)$, the Hilbert space of the vector functions $\alpha = (\alpha_0, \alpha_{1/2}, \alpha_1, \ldots \alpha_N)$ with the scalar product:

$$(\alpha, \beta) = \sum_{i=0}^{2N}\int_0^1 \Delta z_{i/2}\alpha_{i/2}\beta_{i/2}\, d\mu \qquad (30)$$

and the norm:

$$\|\alpha\| = \sqrt{(\alpha,\alpha)},\ \alpha \in M_h(0,2N)\,. \qquad (31)$$

We define the vector functions:

$$\begin{aligned} \varphi &= (u_0, v_{1/2}, u_1, \ldots, u_{N-1}, v_{N-1/2}, u_N) \\ f &= (g_0, r_{1/2}, g_1, \ldots, g_{N-1}, r_{N-1/2}, g_N) \\ \varphi^0 &= (u_0^0, v_{1/2}^0, u_1^0, \ldots, u_{N-1}^0, v_{N-1/2}^0, u_N^0) \end{aligned} \qquad (32)$$

and the operator: $A = L - S$, where

$$L = \begin{bmatrix} \frac{\mu}{\Delta z_0} + \sigma_0 & \frac{\mu}{\sigma_0} & 0 & 0 & \Lambda & 0 & 0 \\ -\frac{\mu}{\Delta z_{1/2}} & \sigma_{1/2} & \frac{\mu}{\Delta z_{1/2}} & 0 & \Lambda & 0 & 0 \\ \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\ 0 & 0 & 0 & 0 & \Lambda & \sigma_{N-1/2} & \frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & 0 & \Lambda & -\frac{\mu}{\Delta z_N} & \frac{\mu}{\Delta z_N} + \sigma_N \end{bmatrix}$$

$$(33)$$

$$S = diag\left(\sigma_{s_{i/2}} \int_0^1 \gamma_{i/2}\, d\mu\right), i = 0,1,...,2N;$$
$$\gamma_{i/2} = \begin{cases} 1 & \text{if } \ i/2 \ \text{is entier number} \\ 0 & \text{if } \ i/2 \ \text{is rational number.} \end{cases} \tag{34}$$

Then, the system (27) – (29) has the form:

$$\frac{d\varphi}{dt} + A\varphi = f, \quad t \in [0,T]$$
$$\varphi(z,\mu,0) = \varphi^0 \tag{35}$$

where φ, f, φ^0 were defined by (32). For simplicity of the writing, we use the same notation as (1) – (3).

We shall prove that L and A are the positive operators. Indeed, let $w \in M$ and then

$$\begin{aligned}(Lw,w) &= \int_0^1 \Delta z_0 \left(\mu \frac{w_{1/2} + w_0}{\Delta z_0} + \sigma_0 w_0\right) w_0 d\mu + \\ &+ \int_0^1 \Delta z_{1/2} \left(\mu \frac{w_1 - w_0}{\Delta z_{1/2}} + \sigma_0 w_0\right) w_{1/2} d\mu + \\ &+ \sum_{i=1}^{N-1} \int_0^1 \Delta z_i \left(\mu \frac{w_{i+1/2} + w_{i-1/2}}{\Delta z_i} + \sigma_i w_i\right) w_i d\mu + \\ &+ \sum_{i=1}^{N-1} \int_0^1 \Delta z_{i+1/2} \left(\mu \frac{w_{i+1} - w_i}{\Delta z_{i+1/2}} + \sigma_{i+1/2} w_{i+1/2}\right) w_{i+1/2} d\mu + \\ &+ \int_0^1 \Delta z_N \left(\mu \frac{w_N + w_{N-1}}{\Delta z_N} + \sigma_N w_N\right) w_N d\mu = \\ &= \int_0^1 \mu (w_0^2 + w_N^2) d\mu + \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2} \sigma_{i/2} w_{i/2}^2 \ d\mu > \\ &> \sigma_0 \sum_{i=1}^{2N} \int_0^1 \Delta z_{i/2} w_{i/2}^2 d\mu = \sigma_0 \| w \|^2 > 0\end{aligned}$$

because continuous functions on $[0,H]$,σ_i, are bounded on $[0,H]$ and by hypothesis $\sigma \geq \sigma_0 > 0$. Using above results, we obtain:

$$\begin{aligned}(Aw,w) &== \int_0^1 \mu (w_0^2 + w_N^2) d\mu + \\ &+ \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2} (\sigma_{i/2} w_{i/2}^2 - \sigma_{s_{i/2}} w_{i/2} \int_0^1 w_{i/2} \gamma_{i/2} d\mu') d\mu \\ &> \int_0^1 \mu (w_0^2 + w_N^2) d\mu + \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2} w_{i/2}^2 (\sigma_{i/2} - \sigma_{s_{i/2}}) d\mu \\ &> \sigma_{c0} \| w \|^2 > 0.\end{aligned}$$

Now, we find an aprioristic estimation of the solution of the problem (35), using the Cauchy-Schwarz inequality and the property that A is positive. Multiplying scalar (35) by φ and integrating with respect to t, we get

$$\begin{aligned}&\frac{1}{2}\|\varphi\|^2 + \int_0^t (A\varphi,\varphi)dt' = \int_0^t (f,\varphi)dt' + \frac{1}{2}\|\varphi^0\|^2 \leq \\ &\leq \left(\int_0^t \|f\|^2 dt'\right)^{1/2} \left(\int_0^t \|\varphi\|^2 dt'\right)^{1/2} + \frac{1}{2}\|\varphi^0\|^2 \leq \\ &\leq C\left(\int_0^t \|f\|^2 dt'\right)^{1/2} \left(\int_0^t (A\varphi,\varphi) dt'\right)^{1/2} + \frac{1}{2}\|\varphi^0\|^2 \leq \\ &\leq C\left(\int_0^t \|f\|^2 dt'\right)^{1/2} \left(\int_0^t (A\varphi,\varphi) dt' + \frac{1}{2}\|\varphi\|^2\right)^{1/2} + \\ &+ \frac{1}{2}\|\varphi^0\|^2.\end{aligned}$$

Using the following inequality

$$|ab| \leq \frac{1}{4\varepsilon} a^2 + \varepsilon b^2, \varepsilon > 0$$

we find the estimation of the solution φ of the form:

$$\|\varphi\|^2 + \int_0^t (A\varphi,\varphi)dt' \leq K\left(\int_0^t \|f\|^2 dt' + \|\varphi^0\|^2\right)$$

where the constant K does not depend on t and φ.

In the second stage a difference scheme is given to approximate the time derivatives. This is used together with the bicycle splitting-up method,[5], which writes the operator A as a sum:

$$A(t) = A_1(t) + A_2(t), \ A_1(t) \geq 0, A_2(t) \geq 0 \tag{36}$$

Let us divide the close interval $[0, T]$ into n subintervals by choosing points: $t_0 = 0, t_1, \ldots, t_n = T$. Next, we take an arbitrary subinterval: $[t_{j-1}, t_{j+1}] = [t_{j-1}, t_{j-1/2}] \cup [t_{j-1/2}, t_j] \cup [t_j, t_{j+1/2}] \cup [t_{j+1/2}, t_{j+1}]$, which has the length equal to 5τ, where τ is the time step. Approximating the operators A_1, A_2 on this the subinterval by: $\Lambda_k^j = A_k(t_j), k = 1,2$, we shall obtain from (35) a difference system using the Krank-Nicholson scheme, [5]:

$$\frac{\varphi^{j-2/3} - \varphi^{j-1}}{\tau} + \Lambda_1^j \frac{\varphi^{j-1} + \varphi^{j-2/3}}{2} = 0$$

$$\frac{\varphi^{j-1/3} - \varphi^{j-2/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j-2/3} + \varphi^{j-1/3}}{2} = 0$$

$$\frac{\varphi^{j-2/3} - \varphi^{j-1}}{2\tau} = f^j \qquad (37)$$

$$\frac{\varphi^{j+2/3} - \varphi^{j+1/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j+2/3} + \varphi^{j+1/3}}{2} = 0$$
$$\frac{\varphi^{j+1} - \varphi^{j+2/3}}{\tau} + \Lambda_1^j \frac{\varphi^{j+1} + \varphi^{j+2/3}}{2} = 0$$

where f^j is the vector with the components:

$$f_i^j = \frac{1}{t_{j+1} - t_{j-1}} \int_{t_{j-1}}^{t_{j+1}} f_i \, dt, \quad t_j = j\tau$$

In other form we obtain:

$$\begin{aligned}
\varphi^{j-2/3} &= \left(E + \frac{\tau}{2}\Lambda_1^j\right)^{-1}\left(E - \frac{\tau}{2}\Lambda_1^j\right) = T_1^j \varphi^{j-1} \\
\varphi^{j-1/3} &= T_2^j \varphi^{j-2/3} \\
\varphi^{j+1/3} &= \varphi^{j-1/3} + 2\tau f^j \\
\varphi^{j+2/3} &= T_2^j \varphi^{j+1/3} \\
\varphi^{j+1} &= T_1^j \varphi^{j+2/3}
\end{aligned} \qquad (38)$$

where E is unit matrix and :

$$T_k^j = (E + \frac{\tau}{2}\Lambda_k^j)^{-1}(E - \frac{\tau}{2}\Lambda_k^j) \qquad (39)$$

Finally, we have the recurrence formula

$$\begin{aligned}
\varphi^{j+1} &= T_1^j T_2^j T_2^j T_1^j \varphi^{j-1} + 2\tau T_1^j T_2^j f^j = \\
&= T^j \varphi^{j-1} + 2\tau T_1^j T_2^j f^j
\end{aligned} \qquad (40)$$

where $T^j = T_1^j T_2^j T_2^j T_1^j$.

Now, we prove that numerical solution approximates the exact solution with an accuracy of second order in time step τ.

Approximation

For the estimation of approximation order, we shall expand with respect to the power of small τ, the expression

$$\begin{aligned}
T_k^j &= \left(E + \frac{\tau}{2}\Lambda_k^j\right)^{-1}\left(E - \frac{\tau}{2}\Lambda_k^j\right) = \\
&= E - \tau\Lambda_k^j + \frac{\tau^2}{2}(\Lambda_k^j)^2 \ldots
\end{aligned} \qquad (41)$$

when $\frac{\tau}{2}\left\|\Lambda_k^j\right\| < 1,\ k = 1,2.$ Then

$$T_1^j T_2^j = E - \tau\Lambda^j + $$
$$+ \frac{\tau^2}{2}\left[\left(\Lambda_1^j + \Lambda_2^j\right)^2 + (\Lambda_1^j\Lambda_2^j - \Lambda_2^j\Lambda_1^j)\right] + o(\tau^3)$$

where $\Lambda^j = (\Lambda_1^j + \Lambda_2^j)/2$.

If $\Lambda_1^j\Lambda_2^j = \Lambda_2^j\Lambda_1^j$, we get

$$T_1^j T_2^j = E - \tau\Lambda^j + \frac{\tau^2}{2}(\Lambda^j)^2 + o(\tau^3) \qquad (42)$$

When the operators are non-commutative, the approximation with the splitting-up algorithm is of the first order with respect to τ.

Let us now consider

$$\begin{aligned}
T^j &= \prod_{k=1}^{2} T_k^j \prod_{k=2}^{1} T_k^j = T_1^j T_2^j T_2^j T_1^j = \\
&= E - 2\tau\Lambda^j + \frac{(2\tau)^2}{2}(\Lambda^j)^2 + o(\tau^3)
\end{aligned} \qquad (43)$$

Hence, the following estimation is valid in the interval $[t_{j-1}, t_{j+1}]$:

$$\begin{aligned}
\varphi^{j+1} &= \left[E - 2\tau\Lambda^j + \frac{(2\tau)^2}{2}(\Lambda^j)^2\right]\varphi^{j-1} + \\
&+ 2\tau(E - \tau\Lambda^j) f^j + o(\tau^3)
\end{aligned} \qquad (44)$$

and

$$\frac{\varphi^{j+1} - \varphi^{j-1}}{2\tau} + \Lambda^j(E - \tau\Lambda^j)\varphi^{j-1} = (E - \tau\Lambda^j) f^j + O_1(\tau^2)$$

Using the Taylor series expansion of the solution φ in the neighborhood of the point t_{j-1} and substituting t_j for t, we can write:

$$\varphi^j = \varphi(z, \mu, t_j) = \varphi^{j-1} + \left(\frac{\partial\varphi}{\partial t}\right)^{j-1}\tau + o(\tau^2) \qquad (45)$$

Then, we eliminate $\left(\frac{\partial\varphi}{\partial t}\right)^{j-1}$, writing the transport equation (35) in the point t_{j-1} in the form:

$$\left(\frac{\partial\varphi}{\partial t}\right)^{j-1} = -\Lambda^j\varphi^{j-1} + f^j + o_2(\tau)$$

and (45) becomes:

$$\varphi^j = \varphi^{j-1}(E - \tau\Lambda^j) + \tau f^j + o(\tau^2)\,.$$

This relation is an approximation with the accuracy of second order in time step τ of the initial equation (35) on the interval $[t_{j-1}, t_{j+1}]$.

Finally, we get

$$\frac{\varphi^{j+1} - \varphi^j}{2\tau} + \Lambda^j\varphi^j = f^j + o(\tau^2) \qquad (46)$$

To find the solution of the system (37), we consider the first equation, the second and the fourth equations for a fixed μ and the operators (33) and (34). We get

$$\begin{bmatrix} 1+\frac{\tau}{2}\frac{\mu}{\Delta z_0} & \frac{\tau}{2}\frac{\mu}{\Delta z_0} & 0 & \ldots & 0 & 0 & 0 \\ -\frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & 1 & \frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & \ldots & 0 & 0 & 0 \\ 0 & -\frac{\tau}{2}\frac{\mu}{\Delta z_1} & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -\frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} & 1 & \frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \ldots & 0 & -\frac{\tau}{2}\frac{\mu}{\Delta z_N} & 1+\frac{\tau}{2}\frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \mathrm{M} \\ v_{N-1/2}^{j-2/3} \\ u_N^{j-2/3} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\frac{\tau}{2}\frac{\mu}{\Delta z_0} & -\frac{\tau}{2}\frac{\mu}{\Delta z_0} & 0 & \ldots & 0 & 0 & 0 \\ \frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & 1 & -\frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & \ldots & 0 & 0 & 0 \\ 0 & \frac{\tau}{2}\frac{\mu}{\Delta z_1} & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} & 1 & -\frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \ldots & 0 & \frac{\tau}{2}\frac{\mu}{\Delta z_N} & 1-\frac{\tau}{2}\frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-1} \\ v_{1/2}^{j-1} \\ u_1^{j-1} \\ \mathrm{M} \\ v_{N-1/2}^{j-1} \\ u_N^{j-1} \end{bmatrix}$$

.

$$\begin{bmatrix} 1+\frac{\tau}{2}\left(\sigma_0 - \sigma_{s0}\int_0^1 d\mu\right) & 0 & 0 & \Lambda & 0 \\ 0 & 1+\frac{\tau}{2}\sigma_{1/2} & 0 & \Lambda & 0 \\ 0 & 0 & 1+\frac{\tau}{2}\left(\sigma_1 - \sigma_{s1}\int_0^1 d\mu\right) & \Lambda & 0 \\ \Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\ 0 & 0 & 0 & \Lambda & 1+\frac{\tau}{2}\left(\sigma_N - \sigma_{sN}\int_0^1 d\mu\right) \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-1/3} \\ v_{1/2}^{j-1/3} \\ u_1^{j-1/3} \\ \mathrm{M} \\ u_N^{j-1/3} \end{bmatrix} =$$

$$
=\begin{bmatrix}
1-\frac{\tau}{2}\left(\sigma_0-\sigma_{s0}\int\limits_0^1 d\mu\right) & 0 & 0 & \Lambda & 0 \\
0 & 1-\frac{\tau}{2}\sigma_{1/2} & 0 & \Lambda & 0 \\
0 & 0 & 1-\frac{\tau}{2}\left(\sigma_1-\sigma_{s1}\int\limits_0^1 d\mu\right) & \Lambda & 0 \\
\Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\
0 & 0 & 0 & \Lambda & 1-\frac{\tau}{2}\left(\sigma_N-\sigma_{sN}\int\limits_0^1 d\mu\right)
\end{bmatrix}
\cdot
\begin{bmatrix}
u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \mathrm{M} \\ u_N^{j-2/3}
\end{bmatrix}
$$

We obtain the following relations for the numerical solution:

$$
\varphi^{j-1/3}=\left(E+\frac{\tau}{2}A_2^j\right)^{-1}\left(E-\frac{\tau}{2}A_2^j\right)\varphi^{j-2/3}
$$

$$
\varphi^{j-1/3}=\begin{bmatrix}
\frac{1}{1+\sigma_{c0}\tau/2} & 0 & \ldots & 0 \\
0 & \frac{1}{1+\sigma_{1/2}\tau/2} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \frac{1}{1+\sigma_{cN}\tau/2}
\end{bmatrix}\cdot
$$

$$
\cdot\begin{bmatrix}
\left(1-\frac{\sigma_0\tau}{2}\right)u_0^{j-2/3}+\frac{\sigma_{s0}\tau}{2}\int\limits_0^1 u_0^{j-2/3}d\mu & 0 & \ldots. & 0 \\
0 & \left(1-\frac{\tau\sigma_{1/2}}{2}\right)v_{1/2}^{j-2/3} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots. \\
0 & 0 & \ldots & \left(1-\frac{\sigma_N\tau}{2}\right)u_N^{j-2/3}+\frac{\sigma_{sN}\tau}{2}\int\limits_0^1 u_N^{j-2/3}d\mu
\end{bmatrix}
$$

Elements of the product are:

$$
\varphi_i^{j-1/3}=\frac{1}{1+\frac{\tau\sigma_{ci}}{2}}\left[\left(1-\frac{\tau\sigma_i}{2}\right)\varphi_i^{j-2/3}+\frac{\tau\sigma_{si}}{2}\int\limits_0^1\varphi_i^{j-2/3}d\mu\right], i=0,1,\ldots,N \qquad (47)
$$

$$
\varphi_{i-1/2}^{j-1/3}=\frac{1-\sigma_{i-1/2}\tau/2}{1+\sigma_{i-1/2}\tau/2}\cdot\varphi_i^{j-2/3},\ \ i=1,2,\ldots,N \qquad (48)
$$

Analogously, we have

Proceedings of the 5th WSEAS Int. Conf. on Artificial Intelligence, Knowledge Engineering and Data Bases, Madrid, Spain, February 15-17, 2006 (pp106-116)

$$\varphi_i^{j+2/3} = \frac{1}{1+\frac{\tau\sigma_{ci}}{2}} \cdot$$

$$\cdot\left[\left(1-\frac{\tau\sigma_i}{2}\right)\varphi_i^{j+1/3} + \frac{\tau\sigma_{si}}{2}\int_0^1 \varphi_i^{j+1/3} d\mu\right], \quad (49)$$

$$i = 0, 1, \ldots, N$$

$$\varphi_{i-1/2}^{j+2/3} = \frac{1-\sigma_{i-1/2}\tau/2}{1+\sigma_{i-1/2}\tau/2}\cdot\varphi_{i-1/2}^{j+1/3}, \; i = 1, 2, \ldots, N \quad (50)$$

At the third stage, we consider the points: $\mu_0 = 0$, μ_1, ..., $\mu_m = 1$, in the interval [0, 1] and compute the integrals with respect to μ, using a numerical integration (trapezoidal approximation):

$$\int_0^1 \psi(\mu)d\mu \approx \sum_{l=1}^{m} S_l \psi_l , \quad \psi_l = \psi(\mu_l) \quad (51)$$

Then, the system (37) can be written in the form:

$$(E + \frac{\tau}{2} A_{1,l}^j)\,\varphi_l^{j-2/3} = (E - \frac{\tau}{2} A_{1,l}^j)\,\varphi_l^{j-1}$$

$$(E + \frac{\tau}{2} A_{2,l}^j)\,\varphi_l^{j-1/3} = (E - \frac{\tau}{2} A_{2,l}^j)\,\varphi_l^{j-2/3}$$

$$\varphi_l^{j+1/3} = \varphi_l^{j-1/3} + 2\tau f_l^{\,j} \quad (52)$$

$$(E + \frac{\tau}{2} A_{2,l}^j)\,\varphi_l^{j+2/3} = (E - \frac{\tau}{2} A_{2,l}^j)\,\varphi_l^{j+1/3}$$

$$(E + \frac{\tau}{2} A_{1,l}^j)\,\varphi_l^{j+1} = (E - \frac{\tau}{2} A_{1,l}^j)\,\varphi_l^{j+2/3}$$

In this choice of the steps, which correspond to the variables z, t , we use the condition:

$$\tau \le \min_i(\Delta z_{i/2}) \quad (53)$$

3 Numerical example

We wish to find the solution of the problem (35):

$$\frac{d\varphi(z,\mu,t)}{dt} + A\,\varphi(z,\mu,t) = f(z,\mu,t),$$

$(z,\mu,0) \in [0,4] \times [0,1] \times [0,2]$.

$$\varphi(z,\mu,0) = \varphi^0(z,\mu)$$

Let us consider the partition of [0,4] into four subintervals of equal length by points:

$$z_0 = 0 < z_{1/2} < z_1 < z_{3/2} < z_2 = 4$$

with:

$\Delta z_0 = z_{1/2} - z_0 = 1$; $\Delta z_{1/2} = z_1 - z_0 = 2$;
$\Delta z_1 = z_{3/2} - z_{1/2} = 2$;
$\Delta z_{3/2} = z_2 - z_1 = 2$; $\Delta z_2 = z_2 - z_{3/2} = 1$.

The partition of the interval [0, 1] is:

$$\mu_0 = 0 < \mu_1 = 1/2 < \mu_2 = 1.$$

For the variable t, we consider the regular partition of the interval [0,2] by the points:
$t_0 = 0 < t_{1/3} < t_{2/3} < t_1 < t_{4/3} < t_{5/3} < t_2 = 2$.
The initial value problem is defined by:

$$\varphi^0 = \left(u_0^0, v_{1/2}^0, u_1^0, v_{3/2}^0, u_2^0\right) = \left(1,1,1,1,1\right)$$

The functions $\sigma(z)$, $\sigma_s(z)$ and f, which here depends only of μ are defined with the help of the fig.1 and fig.2. The values of $\varphi_i, i \in \{0, 1/2, 1, 3/2, 2\}$ with respect to μ_l and t_j are presented in table 1.

From the relations (8) and using the mean values for $u_{1/2}$, v_1, $u_{3/2}$ we obtain the density, φ^+, for $\mu > 0$ and the density, φ^-, for $\mu < 0$ for each value of z_i and t_j :

$$\varphi^{+}(1/2,\mu,t)=v_{1/2}+\frac{u_0+2\cdot u_1}{3} \qquad \varphi^{-}(0,-\mu,t)=2u_0,\ \varphi^{-}(2,-\mu,t)=0$$

$$\varphi^{+}(1,\mu,t)=u_1+\frac{v_{1/2}+v_{3/2}}{2} \qquad \varphi^{-}(1/2,\mu,t)=\frac{u_0+2\cdot u_1}{3}-v_{1/2}$$

$$\varphi^{+}(3/2,\mu,t)=v_{3/2}+\frac{2\cdot u_1+u_2}{3} \qquad \varphi^{-}(1,\mu,t)=u_1-\frac{v_{1/2}+v_{3/2}}{2}$$

$$\varphi^{+}(2,\mu,t)=2u_N,\quad \varphi^{+}(0,\mu,t)=0 \qquad \varphi^{-}(3/2,\mu,t)=\frac{2\cdot u_1+u_2}{3}-v_{3/2}$$

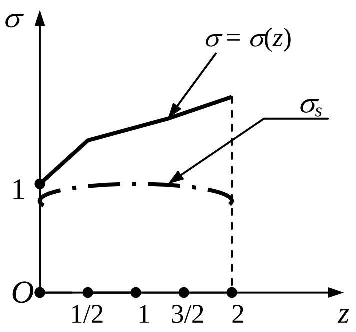

Fig. 1

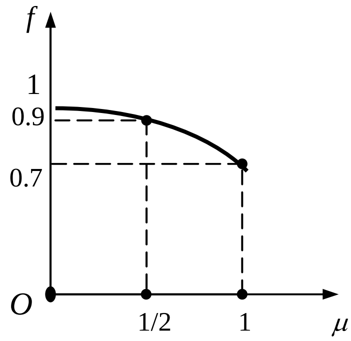

Fig. 2

Table 1

φ	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=1/2$	$\mu=0$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$
u_0	1	0.69	0.44	0.92	0.67	0.46	1.25	0.97	0.69	1.17	0.94	0.71	1.17	0.73	0.38
$v_{1/2}$	1	0.98	0.95	0.62	0.62	0.59	0.95	0.9	0.82	0.59	0.56	0.5	0.59	0.54	0.43
u_1	1	0.99	0.99	0.87	0.85	0.87	1.2	1.17	1.09	1.05	1.03	0.98	1.05	1.04	0.98
$v_{3/2}$	1	1.	1.	0.56	0.56	0.56	0.89	0.86	0.79	0.5	0.48	0.44	0.5	0.5	0.5
u_2	1	1.	1.	0.69	0.69	0.69	0.99	0.99	0.92	0.68	0.68	0.64	0.68	0.65	0.59

Table 2

φ +	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1/2	2	1.88	1.76	1.52	1.4	1.32	2.17	2	1.77	1.7	1.56	1.35	1.7	1.47	1.2
1	2	1.99	1.97	1.5	1.44	1.44	2.12	2.05	1.9	1.6	1.55	1.45	1.6	1.6	1.45
3/2	2	2	2	1.37	1.36	1.37	2	1.97	1.83	1.4	1.39	1.31	1.4	1.4	1.35
2	2	2	2	1.38	1.38	1.38	1.98	1.98	1.84	1.36	1.36	1.3	1.36	1.3	1.2

Table 3

φ -	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$	$\mu=0$	$\mu=1/2$	$\mu=1$
0	2	1.4	0.87	1.85	1.34	0.92	2.5	1.94	1.38	2.34	1.88	1.42	2.34	1.46	1.96
1/2	0	0	0	0.27	0.18	0.14	0.27	0.2	0.14	0.5	0.44	0.39	0.5	0.29	0.35
1	0	0.4	0.02	0.28	0.26	0.29	0.3	0.28	0.29	0.51	0.51	0.51	0.51	0.52	0.52
3/2	0	0.06	0	0.25	0.24	0.24	0.24	0.25	0.25	0.41	0.43	0.43	0.42	0.4	0.35
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

It has been shown above, when $\mu > 0$, the density φ^+ has a maximum value in $t = 4/3$ and a minimum value in $t = 2/3$. When $\mu < 0$, the density φ^- has the same value as φ^+ in $z = 1$ and $t = 2/3$, but in $t = 1/3$, $\varphi^- \approx 0$. Also, we remark that the density φ^- and φ^+ increase when μ decrease.

The results of this numerical example prove its practical importance: how depends the density in a point z at the time t for different values of angle ν.

References:

[1]. Ackroyd R.T., Gashut A.M., O.A. Abuzid O., Discontinuous Variational Solutions for the Neutron Diffusion Equation, *Ann. Nucl. Energy*, **23**, (1996), 1289.

[2]. Case K. M. and Zweifel P. F., *Linear Transport Theory*, Addison-Wesley, Massachusetts, 1967.

[3]. Davis W. R., *Classical Fields, Particles and the Theory of Relativity* , Gordon and Breach, New York, 1970.

[4]. Glasstone S. and Kilton C., *The Elements of Nuclear Reactors Theory*, Van Nostrand, Toronto – New York – London, 1982.

[5]. Marchouk G., *Méthodes de Calcul Numérique*, Édition MIR de Moscou, 1980.

[6]. Marchouk G. and Shaydourov V., *Raffinement des Solutions des Schémas aux Différences*, Édition MIR de Moscou, 1983.

[7]. Marciuk G. and Lebedev V. , *Cislennie Metodî v Teorii Terenosa Neitronov*, Atomizdat, Moscova, 1971.

[8]. Martin O., Une Méthode de Résolution de l'Équation du Transfert des Neutrons, *Rev.Roum. Sci. Tech.- Méc. Appl.*, **37**,6, (1992), 623-646.

[9] Martin O., Analytical and Numerical Solutions for One- Dimensional Transport Equation, *WSEAS TRANSACTIONS on CIRCUITS and SYSTEMS*, **3**,7(2004), 383-389.

[10]. Mihailescu N., Oscillations in the Power Distribution in a Reactor, *Rev. Nuclear Energy*, **9,** 1-4 (1998), 37-41.

[11]. Mokhtar-Kharroubi M., *Topics in Neutron Transport Theory*, Series on Advances in Math. For Applied Sciences, Bellomo and Brezzi, 1997.

[12]. Ntouyas S. K., *Global Existence for Neutral Functional Integral Differential Equations*, Proc. 2nd World Congress of Nonlinear Analysts, 1997, 2133-2142.

[13]. Parton V.Z., Perlin P.I., *Mathematical Methods o f the Theory of Elasticity*, MIR Publishers Moscow,1984.

[14]. Pilkuhn H., *Relativistic Particle Physics*, Springer Verlag, New York - Heidelberg-Berlin, 1980.

[15]. Rahnema F., Ravetto P., On the Equivalence on Boundary and boundary Condition Perturbations in Transport Theory, *Nuclear Sci. Eng.*, **128** (1998), 209-223.

[16]. Weston M., Stacey M., *Nuclear Reactor Physics*, Wiley, New York, 2001.

[17]. Julia Bondarenko, NikosMastorakis,"Solution of Partial Differential Equation for the Option Price by Means the Modified Parametric Method", *WSEAS Trans. On Mathematics*, Issue 4, Volume 3, October 2004, pp.737-743.

[18]. Nikos Mastorakis, "Numerical Solution of Non-Linear Ordinary Differential Equations via Collocation Method, *WSEAS TRANS. on INFORMATION SCIENCE & APPLICATIONS*, Issue 5, Volume 2, May 2005, pp.467-473.

[19]. Nikos Mastorakis, Olga Martin, About the Numerical Solution of a Stationary Transport Equation", *WSEAS TRANS. on INFORMATION SCIENCE & APPLICATIONS*, Issue 9, Volume 2, September 2005, pp.1373-1380.