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A Study on Normalized LMS Algorithm Using Refined Filtering Technique

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Abstract: - We investigate the convergence behavior of the normalized least mean square (NLMS) algorithm in the structure of a linear transversal filter. At the n-th iteration, the traditional NLMS transversal filter generates the n-th output signal by using linear convolution of the n-th input vector and the n-th coefficient vector. Based on this result, the n-th coefficient vector is updated to the $n+1$-th coefficient vector. We attempt a refined filtering (RF) approach to the NLMS transversal filter, to generate another output signal by linear convolution of the n-th input vector and the $n+1$-th coefficient vector. Theoretical analysis and computer simulation demonstrate the effectiveness of the RF technique.

Key-Words: - linear transversal filter, normalized LMS algorithm, adaptive line enhancer, refined filtering

1 Introduction

For the last several decades, adaptive filtering techniques have attracted considerable attention in many application areas of signal processing. As the filter structure, a transversal filter has been used due to the simplicity of its structure. As the adaptation technique of the coefficients, the least mean square (LMS) algorithm [1] is the most popular due to its efficiency and robustness against numerical computation.

The normalized LMS (NLMS) algorithm is one of the modified versions of the LMS algorithm. The NLMS adaptation scheme is implemented as follows:

$$y_n = X_n^T C_n, \tag{1}$$

$$e_n = d_n - y_n, \tag{2}$$

$$C_{n+1} = C_n + \nabla_n \tag{3}$$

$$= C_n + \frac{\mu}{X_n^T X_n + \beta} e_n X_n, \tag{4}$$

where $X_n = [x_n, x_{n-1}, \ldots, x_{n-M+1}]^T$ is the input vector at the n-th iteration, T denotes transpose, $C_n = [c_0(n), c_1(n), \ldots, c_{M-1}(n)]^T$ is the filter coefficient vector, y_n is the output signal, e_n is the error signal, d_n is the desired signal, μ is the step-size to control the convergence, and β is a small positive real constant to avoid division by zero. It is known that advantages of the NLMS algorithm over the LMS algorithm are mainly potentially-faster speed of convergence [2] and stability for the following known range of the step-size:

$$0 < \mu < 2. \tag{5}$$

The NLMS algorithm requires a small increase of computation to the LMS algorithm. Thus, the NLMS algorithm is often used rather than the LMS algorithm and several modified NLMS algorithms have been introduced [3, 4, 5].

It is also known that for $\mu = 1$ and $\beta = 0$, the NLMS algorithm works to minimize $\|C_{n+1} - C_n\|^2$ where $\|\cdot\|^2$ denotes the squared Euclidean norm, subject to the constraint [1]:

$$d_n = X_n^T C_{n+1}. \tag{6}$$

Using the updated coefficient vector and the current input vector, we consider to obtain another output and error signals as

$$y_{pn} = X_n^T C_{n+1}, \tag{7}$$

$$e_{pn} = d_n - y_{pn}, \tag{8}$$

respectively, after the traditional operations of (1)-(4). We call the operations of (7) and (8) the refined filtering (RF) technique in this paper. From the constraint of (6),

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e_{pn} is obviously equivalent to zero for $\mu = 1$ and $\beta = 0$. However, no papers have reported the behavior of e_{pn} for $\mu \neq 1$.

In this paper, we theoretically analyze the performance of the NLMS transversal filter using the RF technique for several settings of μ. We also study an effect of β on the improvement degree provided by using the RF technique.

The remainder of this paper is organized as follows. In Section 2, we clarify the effect of the RF technique. We compare the derived result with some simulation results in Section 3. Section 4 concludes this paper.

2 Analysis

We theoretically show the effectiveness of the RF technique. Substituting (3) into (7) and using (1), we rewrite y_{pn} as

$$\begin{aligned} y_{pn} &= X_n^T(C_n + \nabla_n) \\ &= X_n^T C_n + X_n^T \nabla_n \\ &= y_n + X_n^T \nabla_n. \end{aligned} \tag{9}$$

From (4), we have

$$\begin{aligned} X_n^T \nabla_n &= \frac{\mu}{X_n^T X_n + \beta} e_n X_n^T X_n \\ &= \mu e_n \alpha_n \end{aligned} \tag{10}$$

where

$$\alpha_n = \frac{X_n^T X_n}{X_n^T X_n + \beta}. \tag{11}$$

Substitution of (10) into (9) and use of (2) yield

$$\begin{aligned} y_{pn} &= y_n + \mu e_n \alpha_n \\ &= y_n + \mu \alpha_n (d_n - y_n) \\ &= y_n + \mu \alpha_n d_n - \mu \alpha_n y_n \\ &= \mu \alpha_n d_n + (1 - \mu \alpha_n) y_n. \end{aligned} \tag{12}$$

We here substitute (12) into (8) and use (2). The e_{pn} is then rewritten as

$$\begin{aligned} e_{pn} &= d_n - (\mu \alpha_n d_n + (1 - \mu \alpha_n) y_n) \\ &= d_n - \mu \alpha_n d_n - (1 - \mu \alpha_n) y_n \\ &= (1 - \mu \alpha_n) d_n - (1 - \mu \alpha_n) y_n \\ &= (1 - \mu \alpha_n)(d_n - y_n) \\ &= (1 - \mu \alpha_n) e_n. \end{aligned} \tag{13}$$

From this result, we obtain

$$|e_{pn}|^2 = (1 - \mu \alpha_n)^2 |e_n|^2. \tag{14}$$

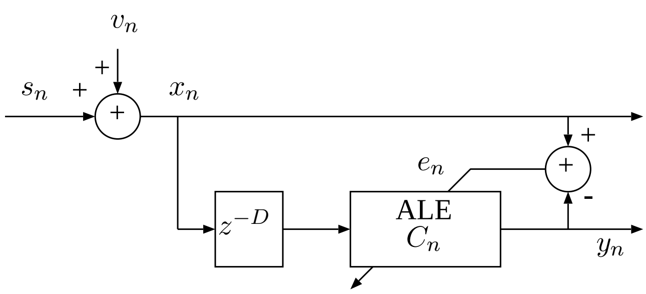

Figure 1: Block diagram of the NLMS transversal enhancer.

Let us assume

$$\alpha_n \approx 1 \tag{15}$$

because β is a small positive constant. Equation (14) is then reexpressed as

$$|e_{pn}|^2 \approx (1 - \mu)^2 |e_n|^2. \tag{16}$$

We here focus on that μ must be set in the range of (5). Then we obtain

$$(1 - \mu)^2 < 1. \tag{17}$$

From (16) and (17), therefore, $|e_{pn}|^2$ is obviously less than $|e_n|^2$.

3 Computer Simulations

We test the NLMS transversal filter algorithm using the RF technique in the scenario of adaptive line enhancement [6].

3.1 Adaptive line enhancement

Figure 1 illustrates the configuration of an adaptive line enhancer (ALE) based on a transversal filter structure. The s_n is a sinusoidal signal, which is mathematically given by

$$s_n = A \sin(2\pi f n) \tag{18}$$

where A and f denote the amplitude and frequency of s_n, respectively. The v_n is a zero mean additive white Gaussian noise with variance σ^2. Therefore, the noisy sinusoidal signal, x_n, is expressed by

$$x_n = s_n + v_n. \tag{19}$$

The ALE uses a delayed noisy sinusoidal signal as the filter input. The input vector at the n-th iteration is therefore expressed by

$$X_n = [x_{n-D}, x_{n-D-1}, \ldots, x_{n-D-M+1}]^T. \tag{20}$$

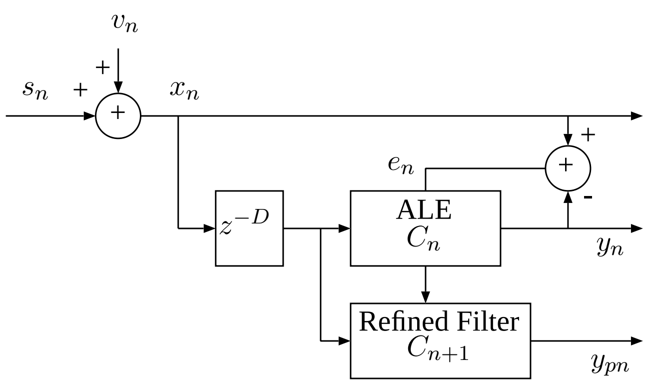

Figure 2: Block diagram of the NLMS transversal enhancer with the RF technique.

where D is a decorrelation delay. We use x_n as the desired signal d_n for the ALE at the n-th iteration, which means

$$d_n = x_n. \tag{21}$$

In the same form as (1)-(4), therefore, the coefficients of the ALE are adapted as follows.

$$\begin{aligned} y_n &= X_n^T C_n, \\ e_n &= x_n - y_n, \\ C_{n+1} &= C_n + \nabla_n \\ &= C_n + \frac{\mu}{X_n^T X_n + \beta} e_n X_n. \end{aligned}$$

Figure 2 shows the configuration of an ALE with the RF technique. The adaptation of the ALE with the RF is implemented in the same as the ALE without RF. We prepare another linear transversal filter for the RF implementation. After the adaptation of the ALE at the n-th iteration, C_{n+1} is copied from the ALE into the refined filter and y_{pn} is calculated. The refined output signal y_{pn} is obtained by

$$y_{pn} = X_n^T C_{n+1}.$$

3.2 Results

We draw the conditions of computer simulations in Table 1. Figure 3 illustrates the convergence of the NLMS ALEs with and without the RF technique. From this figure, the mean square error (MSE) of the NLMS ALEs with the RF technique is obviously smaller than that without the RF technique. Let us confirm the property of (16) discussed in the previous section. The average of MSE of the NLMS ALEs without the RF technique is

Table 1: Conditions in computer simulations.

filter length	$M = 12$
delay	$D = 5$
sampling frequency	$1[Hz]$
frequency of sinusoid	$f = 0.01[Hz]$
amplitude of sinusoid	$A = 1$
noise variance	$\sigma^2 = 1$
stable parameter	$\beta = 0.01$
individual trials	100runs

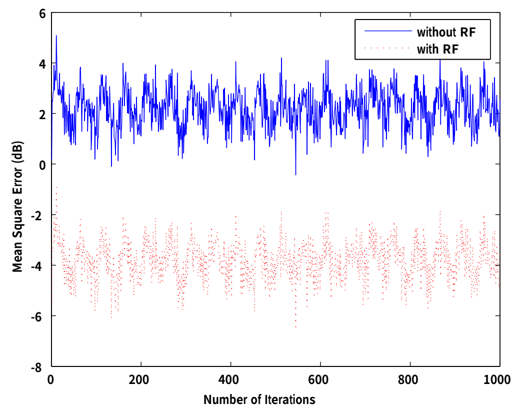

Figure 3: Convergence of the NLMS ALEs for $\mu = 0.5$ with and without the RF technique.

2.2805(dB) and that with the RF technique is -3.7345(dB). Thus, we have

$$|e_n|^2 = 10^{2.2805/10} = 1.6906, \tag{22}$$
$$|e_{pn}|^2 = 10^{-3.7345/10} = 0.4232. \tag{23}$$

The ratio of the two MSEs is then expressed by

$$\frac{|e_{pn}|^2}{|e_n|^2} = \frac{0.4232}{1.6906} = 0.2503. \tag{24}$$

We compare the analysis result (16) with the simulation result (24). We rewrite (16) as

$$\frac{|e_{pn}|^2}{|e_n|^2} \approx (1-\mu)^2. \tag{25}$$

Substituting $\mu = 0.5$ into (25), we have

$$\frac{|e_{pn}|^2}{|e_n|^2} \approx (1-0.5)^2 = 0.25. \tag{26}$$

Comparison of (24) and (26) obviously suggests the effectiveness of the RF technique. From (16), we here describe the degree of improvement provided by the RF technique in the form of dB as

$$\begin{aligned} 10\log_{10}(|e_{pn}|^2) &\approx 10\log_{10}((1-\mu)^2|e_n|^2) \\ &\approx 10\log_{10}((1-\mu)^2) + 10\log_{10}(|e_n|^2). \end{aligned} \tag{27}$$

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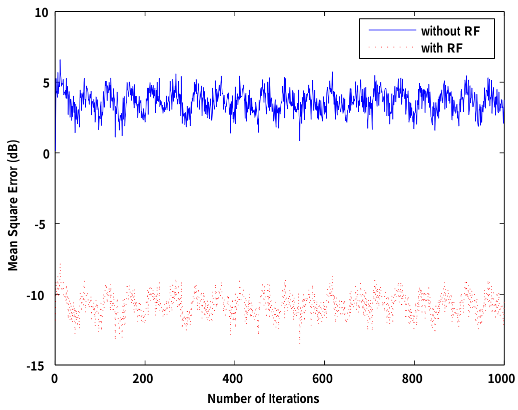

Figure 4: Convergence of the NLMS ALEs for $\mu = 0.8$ with and without the RF technique.

We further substitute $\mu = 0.5$ into (27). Then, the degree of improvement is given by

$$10 \log_{10}((1-\mu)^2) = 10 \log_{10}(0.25) = -6.0206(\text{dB}). \quad (28)$$

Therefore, the setting of $\mu = 0.5$ provides an improvement degree of about -6(dB).

3.3 Effect of β on the improvement degree

We can confirm that there exists a small difference between (24) and (26). The cause of this phenomenon could be α_n, which is affected by β because if $\beta \neq 0$, then $\alpha_n \neq 1$. This is discussed here. As shown by (11), α_n is less than 1. In the case of $0 < \mu \leq 1$, therefore,

$$(1-\mu)^2 < (1-\mu\alpha_n)^2. \quad (29)$$

Similarly, in the case of $1 < \mu < 2$,

$$(1-\mu)^2 > (1-\mu\alpha_n)^2. \quad (30)$$

In Fig.3, $\mu = 0.5$ was used. We thus consider (29) here. The simulation result of Fig.3, (24), does not use the assumption of (15) and therefore is equivalent to $(1-\mu\alpha_n)^2$. The analysis result (26) uses the assumption of (15) and is equivalent to $(1-\mu)^2$. By comparing the two results, we can confirm that $(1-\mu\alpha_n)^2$ is greater than $(1-\mu)^2$ in the case of $0 < \mu \leq 1$.

3.4 Effect of μ on the improvement degree

We finally consider an effect of μ on the degree of improvement given by the RF technique. Figure 4 illustrates the convergence of the two NLMS ALEs where $\mu = 0.8$ is commonly used. From this figure, the MSEs of the NLMS ALEs with and without the RF technique are 3.2974(dB) and -10.6601(dB), respectively. Therefore, the degree of improvement is -13.9574(dB). Comparison of Fig.3 with Fig.4 suggests that when μ is set to a value close to 1, the improvement degree is large. The theoretical analysis result (27) also suggests the effect of μ. Assuming (15), the improvement degree is given by

$$\lim_{\mu \to 1} 10 \log_{10}((1-\mu\alpha_n)^2) \approx -\infty. \quad (31)$$

Hence, we conclude that the improvement degree given by the RF technique is the greatest for the setting of $\mu = 1$. This property is equivalent to the constraint of (6).

4 Conclusion

We have investigated the convergence behavior of the NLMS transversal filter algorithm using the RF technique. Theoretical analysis and computer simulation visualize that the RF technique always provides a performance improvement and that when the step-size is set to 1, the RF technique provides the greatest of the improvement degree.

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