Efficiency universal curves for rectangular fins based on a generalized characteristic length

1ALHAMA, F. and 2LUNA ABAD, J.P.
1Department of Applied Physics, 2Department of Thermal Engineering and Fluids
Polytechnic University of Cartagena
ETSI, Campus Muralla del Mar, Cartagena, MURCIA
SPAIN

Abstract: - A new characteristic length for 2-D rectangular fins, which depends on Biot transversal number, is investigated. This length tends to the classical characteristic length when the transversal Biot number is negligible (1-D hypothesis). Based on the proposed new length, the solutions for temperature fields and heat fluxes in the fin with adiabatic end, as well as the efficiency, can be presented as a function of the dimensionless ratio “actual length/characteristic length”, using the transversal Biot as a parameter. Discriminated dimensional analysis is used to determine the classical characteristic length, while a numerical solution to the fin conduction problem is provided by the network method.

Key-Words: - Characteristic length, discriminated dimensional analysis, 2-D conduction, cylindrical pin fin, efficiency.

1 Introduction

The order of magnitude of the characteristic length for 1-D, isothermal base, large rectangular fins, \(l_g\), can be rapidly obtained, as a “hidden” quantity, by using discriminated dimensional analysis [1]. This technique has already been used to study the equivalent problem in cylindrical pin fins [2,3]. As in that case, an added simple quantitative criterion established a precise value for this characteristic length which can be used to make the actual fin length dimensionless. The characteristic length, \(l_g\), is the order of magnitude of the fin portion where most heat dissipation occurs. More precisely, it is defined as the fin length for which the temperature decreases to a specified fraction of temperature excess, \(\Delta \theta\), the difference between the base and the surrounding temperature. \(l_g\) is the inverse of the known fin parameter, \(m\), which is obtained from the fin conduction equation.

However, when 1-D approximation is not applicable (relatively large Biot transversal numbers, \(B_i\)), the characteristic length depends on \(B_i\), and a new characteristic length, \(l'_g\), arises for each \(B_i\). The assumption of the 1-D hypothesis provides one temperature at each point of the transversal section of the fin, meaning that the specified fraction of \(\Delta \theta\) that defines \(l'_g\) does not depend on the transversal coordinate, (fig. 1). In contrast, with the frequently accepted 2-D hypothesis [4,5] (as occurs in many practical applications, particularly in natural convection), the specified fraction of temperature excess depends on the transversal coordinate, which means that a particular value for the transversal coordinate must be chosen to define the new \(l'_g\). In this study we choose the location \(y=e\) (the semi thickness of the fin).

In short, \(l'_g\) is the portion of a long 2-D fin, for which the surface temperature criterion \(\theta_b \leq \theta_z \leq \theta_b - \Delta \theta\) is satisfied. The kind of boundary condition at the fin end does not influence \(l'_g\) since the fin considered is very large. Within this portion of the fin, \(0 \leq z \leq l'_g\), a good fraction of the heat flux that penetrates at the base is dissipated.

Firstly, we analytically and numerically determine \(l_g\) using a 1-D network model. Secondly, we investigate the dependence \(l'_g = l'_g(B_i)\) by analyzing the profiles of temperature and heat flux at the surface, for long 2-D fins of different \(B_i\). For this purpose, a set of 2-D network models [6] is designed and numerically simulated.

In order to appreciate the role of \(l'_g\), new profiles for the surface temperature of these fins are represented as a function of adimensionless location, \(x/l'_g\). Errors seen in the comparison with 1-D solutions, for each \(B_i\), are tabulated.

Finally, the fin efficiency (the fundamental dimensionless performance coefficient) of short rectangular 2-D fins is numerically solved. The efficiency of these fins is graphed as a function of the dimensionless length \(l/l'_g\) providing universal design curves.
The network simulation method, used in this work as a numerical technique, has been applied in many engineering disciplines [6], since it provides very accurate solutions with negligible computing times. The network model is simulated by the code Pspice [7].

3 Governing equations

Assuming a slender form fin, b >> 2e, the unsteady rectangular isothermal-base fin with adiabatic end, figure 1, is governed by the following equations:

1-D conduction hypothesis

\[ \frac{\partial}{\partial z} \left( k \frac{\partial \theta}{\partial z} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial \theta}{\partial y} \right) = \rho_c \left( \frac{\partial \theta}{\partial t} \right) \]

\( 0 < z < L, \quad 0 < y < e \) \hfill (1b)

2-D conduction hypothesis

\[ \frac{\partial}{\partial z} \left( k \frac{\partial \theta}{\partial z} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial \theta}{\partial y} \right) = \rho_c \left( \frac{\partial \theta}{\partial t} \right) \]

\( 0 < z < L; \quad 0 < y < e \) \hfill (2b)

\[ j = h(\theta - \theta_{ref}) \quad \text{at } y = e; \quad 0 < z < L; \]

\[ \theta = \theta_b, \quad 0 < y < e, \quad z = 0; \]

\[ \theta = \theta_b, \quad 0 < z < L, \quad 0 < y < e, \quad t = 0; \] \hfill (4b)

Equations (1a-b) refer to the heat conduction equation, while equations (2a-3a) and (2b-4b) refer to the boundary conditions. Finally equations (4a) and (5b) refer to initial condition. Since the long fin solution does not depend on the boundary conditions at the tip, an adiabatic condition is always applied to long fins.
4 The numerical method

Detailed information for applying the network simulation method is found in González-Fernández et al [6]. Details of this application in the field of heat transfer in fins is described in Alarcón et al. [8,9].

The basis of the method is to design a network model whose equations are formally equivalent to the finite-difference differential equations arising from the spatial discretization of the mathematical model equations. Time remains as a continuous variable in the model.

Since the material of the fin is assumed to be isotropic and homogeneous, the network 2-D model of the cell (volume element) is quite simple.

- Applying spatial finite differences to equation (1a) or, equivalently, applying heat balance to a rectangular cell of size $\Delta z$, height $\Delta y$ and width $\Delta x=1$ according to the nomenclature of fig. 2 and Fourier law, it is a straightforward task to derive the value of the resistors of each cell [6]. These are:

$$R_{i-\Delta, j} = R_{i+\Delta, j} = \frac{\Delta z}{2k \Delta y \Delta x}$$

$$R_{i,j-\Delta} = R_{i,j+\Delta} = \frac{\Delta y}{2k \Delta z \Delta x}$$

The complete model, fig. 3, is formed by connecting successive cells in series and by adding boundary conditions at the base (constant voltage generators) and at the dissipative surfaces (resistors). Since the problem is steady-state, initial temperature does not need to be considered.

A sufficient number of cells (no fewer than $40 \times 40$) ensures a negligible error in the field temperature and heat fluxes [10]. These unknowns are solved simultaneously and may be read directly by graphs or by data tables.

For all the models run in this work a number of $40 \times 100$ cells was chosen. Once the network model is completed, it is run in a network simulation program. Pspice, the code used for this purpose, incorporates powerful programming rules that permit the multiple simulation of a set of fins of different parameters (length, $B_i$, etc).

Computing times on a PC Pentium 4 is of the order of 3 minutes for the more complex models of this work.

5 The characteristic lengths from dimensional analysis.

As shown in Luna and Alhama [1,2,3], discriminated dimensional analysis (a vectorial version of classical dimensional analysis) leads directly not only to the dimensionless groups that play an independent role in the solution of the problem but, also, to the “hidden” quantities of the problem, if they exist. Nicolas and Alhama [1] proved that, in contrast to classical dimensional analysis, the groups derived from discriminated analysis do indeed play an independent role in the solution.

According to the geometry of the problem, fig. 1, and choosing three lengths to characterize this geometry, one of the (complete) dimensional basis for the problem is

$$\{L_x, L_y, L_z, Q, \theta, T\}$$

Following the explanations of Luna and Alhama, the list of relevant variables for the 2-D, steady-state, “long” fin problem is $k_y$, $k_z$, $h$, $e$, and $\Delta \theta$. The dimensional equations of these variables are:

$$[k_y] = Q \theta^{-1} T^{-1} L_x^{-1} L_z^{-1} L_y$$
\[ k_z = \frac{Q\theta^{-1}T^{-1}L_x^{-1}L_y^{-1}L_z}{}, \]
\[ h = \frac{Q\theta^{-1}T^{-1}L_y^{-1}L_z^{-1}}{}, \]
\[ [S] = L_y L_z, \]
\[ [P] = L_x, \]
\[ [\Delta\theta] = \theta \]

The length of the fin, \( L_0 \), does not form part of the relevant list since the effective length that dissipates heat is assumed to be a small fraction of \( L_0 \). Dimensional exponents of these variables are grouped in Table 1.

<table>
<thead>
<tr>
<th>Base</th>
<th>P</th>
<th>S_1</th>
<th>k_x</th>
<th>k_y</th>
<th>h</th>
<th>\Delta\theta</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_x )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_y )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( L_z )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T )</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Q )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( \theta )</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Dimensional exponents of the variables

The only one (discriminated) dimensionless group (or \( \pi \) monomial) resulting from the application of pi-theorem to the variables of the table is \( \pi_1 = hS/Pk_y = Bi_0 \), the transversal Biot number. However, we can look for a characteristic length parallel to the z axis, namely \( l_g^* \). Introducing this quantity in Table 1, \( [l_g^*] = L_z \), a new dimensionless group appears: \( \pi_2 = l_g^*/(Sk_y/Ph)^{1/2} \). As a consequence, the characteristic length can be expressed in the form

\[ l_g^* = (S/k_y/Ph)^{1/2}f_1(hS/Pk_y) = (S/k_y/Ph)^{1/2}f_2(Bi_0) \]  \hspace{1cm} (6)

where \( f_1 \) and \( f_2 \) denote arbitrary unknown functions. Also, since the group of variables \( (S/k_y/Ph)^{1/2} \) has the dimension of an axial length, namely \( l^* \), we can write the above expression in the forms

\[ l_g^*/l^* = f_1(hS/Pk_y) = f_2(Bi_0) \]  \hspace{1cm} (7)

When the 1-D hypothesis is assumed, the effect of normal conduction is negligible (\( k_y \) does not appear in the relevant list and \( Bi_0 \) cannot be formed). In this case, it is immediately demonstrated that the only characteristic length that discriminated dimensional analysis provides is \( l^* \). Thus \( l^* \) is a hidden (axial) quantity inherent to 1-D fins, regardless of the actual length of the fin, and can be used to make the actual length dimensionless in the graphics, for example, to represent the efficiency. In fact, this is the way followed in most books based on arguments derived from the work with differential equations where \( l^* \) is termed the fin parameter. As can be seen, dimensional analysis in its discriminated version deals also leads to the same result.

For short fins with adiabatic end, the actual length, \( L_o \), causes a redistribution of temperature profiles and heat fluxes. The introduction of \( L_o \) in the relevant list of Table 1 ([L_o] = L_z) provides two dimensionless groups, \( \pi_1 = hS/Pk_y = Bi_0 \) and \( \pi_2 = l_g^*/(S/k_y/Ph)^{1/2} = L_o/l^* \), on which dimensionless performance coefficients, such as efficiency, will depend.

### 6 Temperature and heat flux profiles

Fig. 4 shows the temperature profile of a 1-D long rectangular fin (\( Bi_0 = 1E-5 \)) obtained numerically and analytically. Both temperature and position are dimensionless, \( (\theta-\theta_{ref})/(\theta_{b}-\theta_{ref}) \) and \( z/l^* \), respectively. The curves are nearly the same. For comparison, the profile for a 1-D long cylindrical spine [2] is presented.

![Fig. 4. Temperature profiles for long 1-D rectangular fins.](image-url)
fallen to 38.6% (14.2% for length 2l, 5.2% for 3l and 1.9% for 3l of the temperature excess, or the length in which total dissipated heat is 64.9% (87.1% for length 2l, 95.3% for 3l and 98.3% for 3l of the total dissipated heat.

The generalized characteristic length

According to the results of discriminated dimensional analysis theory, \( l_g = f(B_i) \), temperature and heat flux profiles for long rectangular, 2-D, fins will depend on \( B_i \). Fig. 6 shows the temperature profiles, at the surface, \( y=e/2 \), (obtained from the numerical solution of the network model) for a set of long fins of \( B_i = 0.001, 0.01, 0.1, 0.2, 0.5, 1 \) and 2. As seen, an increasing in \( B_i \) causes a decrease in the profile since the conduction along \( z \) axis increases.

Extending to 2-D the above definition of \( l^* \) (now named \( l_g^* \)) given for 1-D, Table 2 summarizes the values of \( l_g^* \) for each \( B_i \). S.I. units are used for \( k, h, e \) and \( l^* \) in the Table. Fig. 7 illustrates the dependence of \( l_g^* \) on \( B_i \) (\( l_g^* = f(B_i) \)), and includes the equivalent cylindrical pin fin curve to appreciate the influence of the type of geometry of the fin. Two important aspects derive from fig. 7. On the one hand, a cylindrical geometry is more sensitive to changes in \( l_g^* \) than rectangular geometry. For example, in rectangular fins, \( l^* \) deviates 6.35%, 14.17% and 22.43% from the 1-D value for\( B_i = 0.1, 0.5 \) and 1, while the deviations for cylindrical fins are 7.67%, 18.11% and 27.83%, respectively. On the other hand \( l_g^*/l^* \) tends to 0.95, instead of 1, as \( B_i \) tends to be negligible (\( \approx 1E-4 \)). This is an interesting deviation in relation to the analytical 1-D solution and proves that however small the value of \( B_i \), the effects of 2-D cannot be assumed as negligible.

<table>
<thead>
<tr>
<th>( B_i )</th>
<th>( l^* )</th>
<th>( k )</th>
<th>( h )</th>
<th>( e )</th>
<th>( \frac{l_g^<em>}{l^</em>} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.012649</td>
<td>100</td>
<td>250</td>
<td>0.0004</td>
<td>0.001</td>
<td>0.951</td>
</tr>
<tr>
<td>0.007000</td>
<td>100</td>
<td>1428.6</td>
<td>0.0007</td>
<td>0.01</td>
<td>0.948</td>
</tr>
<tr>
<td>0.003162</td>
<td>100</td>
<td>10000</td>
<td>0.001</td>
<td>0.1</td>
<td>0.937</td>
</tr>
<tr>
<td>0.003354</td>
<td>100</td>
<td>13333.3</td>
<td>0.0015</td>
<td>0.2</td>
<td>0.913</td>
</tr>
<tr>
<td>0.002121</td>
<td>100</td>
<td>33333.3</td>
<td>0.0015</td>
<td>0.5</td>
<td>0.858</td>
</tr>
<tr>
<td>0.002000</td>
<td>100</td>
<td>50000</td>
<td>0.002</td>
<td>1</td>
<td>0.776</td>
</tr>
<tr>
<td>0.002449</td>
<td>100</td>
<td>50000</td>
<td>0.003</td>
<td>1.5</td>
<td>0.702</td>
</tr>
<tr>
<td>0.001768</td>
<td>100</td>
<td>80000</td>
<td>0.0025</td>
<td>2</td>
<td>0.647</td>
</tr>
</tbody>
</table>

Table 2. Generalized characteristic length versus transversal Biot number

Fig. 7. \( l_g^* = l_g^*(B_i) \) for long rectangular fins (universal curve)
from which $B_i$ and $l^*$ can be derived, $B_i = h e / k$ and $l^* = (e k / h)^{1/2}$. Knowing $B_i$, fig. 7 provides $l^*_g$.

**Generalized temperature profiles**

The dimensionless and universal temperature profiles can now be represented as a function of the dimensionless ratio $z/l^*_g$, figure 8, using $B_i$ as a parameter. Note that the horizontal scale refers to a different value of $l^*_g$, which is deduced from the corresponding $B_i$. The influence of $B_i$ is apparent in the profiles. The curves tend to a limit as $B_i$ tends to zero. As expected, the curves intersect at $\theta_{x-e, \text{dimensionless}} = 0.386$, $z/l^*_g = 1$).

![Fig. 8. Dimensionless temperature profiles for long 2-D rectangular fins (universal curves).](image)

The graphs of dimensionless temperature in different sections ($\theta_{x=0}-\theta_{x=e}/\Delta \theta$ as a function of the dimensionless location $z/l^*_g$, for different $B_i$) provide information concerning whether the 1-D assumption (Murray-Gardner hypothesis [8,9]) is appropriate, fig. 9. Certainly, even for small $B_i$ in the order of 0.1, temperature deviations above 5% arise in part of the fin.

As regards the conditions that must be satisfied by a fin to justify the application of the 1-D model, Razelos and Georgiou [11, 12] refer to the parameters $B_i$ (as we do) and to the aspect ratio. However, Krauss [12] asserts that these design criteria, while applicable to many engineering situations, are not applicable in all cases.

![Fig. 9. Profiles of dimensionless temperature difference $\theta_{x=0} - \theta_{x=e}$ for long fins (Universal curves)](image)
7 The efficiency

As is known, this performance and design parameter is defined as the ratio of the actual heat loss to the maximum possible heat loss.

The main disadvantage of this parameter $\eta$, concerns the standard with respect to which the fins are compared to evaluate this coefficient. As a consequence, two fins of different dimensions in the same environment may have the same efficiency but they transmit different quantities of heat. However, despite this disadvantage we will use this sound dimensionless coefficient, to assess the performance of short fins of rectangular profile with adiabatic tip condition and for a reasonable range of $Bi_t$.

Fig. 9 represents the efficiency of these differently sized fins which are isothermal at the base and adiabatic at the tip. The $Bi_t$ numbers cover a reasonable set of values: 0.001, 0.01, 0.1, 0.2, 0.5, 1, 1.5 and 2.

The actual dimensionless lengths, defined in relation to the generalized characteristic length ($l_g^*$) also cover a reasonable range, $0.5 \leq L/l_g^* \leq 3$.

As shown in fig. 10, the efficiency curves intersect one another for fin lengths of around $L \approx 1.84 l_g^*$. Since $l_g^*$ depends on $Bi_t$, the actual $L$ is different for each $Bi_t$.

On the one hand, this means that for short longitudinal fins with $L \approx 1.84 l_g^*$, whatever the value of $Bi_t$, the efficiency value ($\eta \approx 0.54$) does not depend on $Bi_t$. As $L$ diminishes from $1.84 l_g^*$ (increases) from $1.84 l_g^*$, efficiency increases (diminishes). On the other hand, for short longitudinal fins ($L<l_g^*$), efficiency diminishes as $Bi_t$ increases, while for long fins ($L>l_g^*$) efficiency increases as $Bi_t$ increases.

The behavior of the efficiency curves (principally the fact that they cross around a size related with the generalized characteristic length, which, in turn, depends of $Bi_t$) is due to the introduction of $l_g^*$, one of the objectives of this paper. The shape of the efficiency curves would not have been so striking without the use of $l_g^*$.

The way to use the information provided by figure 10 is the same as that already explained for figure 8. Starting from the values of $k$, $h$ and $e$, which determine $Bi_t$ and $l_g^*$ ($Bi_t = he/k$ and $l_g^* = (ke/h)^{1/2}$), and based on the information of fig. 6, which shows the generalized characteristic length, $l_g^*$, fig.10 provides the efficiency of the actual fin of length $L$.

![Fig. 10. Efficiency curves for different Bi values as a function of dimensionless length L/l_g*
(Universal curves)](image)

On the one hand, this means that for short longitudinal fins with $L \approx 1.84 l_g^*$, whatever the value of $Bi_t$, the efficiency value ($\eta \approx 0.54$) does not depend on $Bi_t$. As $L$ diminishes from $1.84 l_g^*$ (increases) from $1.84 l_g^*$, efficiency increases (diminishes). On the other hand, for short longitudinal fins ($L<l_g^*$), efficiency diminishes as $Bi_t$ increases, while for long fins ($L>l_g^*$) efficiency increases as $Bi_t$ increases.

The behavior of the efficiency curves (principally the fact that they cross around a size related with the generalized characteristic length, which, in turn, depends of $Bi_t$) is due to the introduction of $l_g^*$, one of the objectives of this paper. The shape of the efficiency curves would not have been so striking without the use of $l_g^*$.

The way to use the information provided by figure 10 is the same as that already explained for figure 8. Starting from the values of $k$, $h$ and $e$, which determine $Bi_t$ and $l_g^*$ ($Bi_t = he/k$ and $l_g^* = (ke/h)^{1/2}$), and based on the information of fig. 6, which shows the generalized characteristic length, $l_g^*$, fig.10 provides the efficiency of the actual fin of length $L$. 

![Fig. 10. Efficiency curves for different Bi values as a function of dimensionless length L/l_g*
(Universal curves)](image)
Finally, Table 3 shows the absolute errors in efficiency resulting from the comparison of 2-D models (for different $B_i$) and 1-D models (not dependent on $B_i$) for different $B_i$ and for lengths $L=0.5, 1, 1.5, 2, 2.5$ and $3$ times $l_{g^*}$.

It is interesting to see that these errors increase as $B_i$ increases. Also, errors increase as dimensionless fin length both increases or decreases around the central value of $z/l_{g^*}>1.84$. See that errors are always less than 5% for fins with $z/l_{g^*}>1.5$, whatever be $B_i$, while errors are upper than 5% for short fins ($z/l_{g^*}<1$) and $B_i>0.5$.

<table>
<thead>
<tr>
<th>$B_i$</th>
<th>$z/l_{g^*}$</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-3.73</td>
<td>-2.10</td>
<td>-0.68</td>
<td>0.02</td>
<td>0.29</td>
<td>0.37</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>-7.16</td>
<td>-4.45</td>
<td>-1.37</td>
<td>0.30</td>
<td>0.97</td>
<td>1.17</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-10.29</td>
<td>-7.17</td>
<td>-2.29</td>
<td>0.76</td>
<td>2.15</td>
<td>2.64</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>-11.92</td>
<td>-8.84</td>
<td>-2.98</td>
<td>0.9</td>
<td>3.21</td>
<td>4.05</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-13.04</td>
<td>-10.10</td>
<td>-3.82</td>
<td>0.94</td>
<td>3.61</td>
<td>4.85</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Efficiency errors: $100(\eta_{(2-D)} - \eta_{(1-D)})$

8 Conclusions

Firstly, it is demonstrated that discriminated dimensional analysis directly leads to the two dimensionless parameters that provide a complete characterization of isothermal base rectangular long fins and isothermal base rectangular short fins subjected to an adiabatic boundary condition at the tip. These parameters are the transversal Biot number, $B_i$, and the characteristic length, $l^*$. The meaning of $l^*$ can be generalized to a new characteristic length $l_{g^*}$, whose value is a function of $B_i$, which is numerically determined in the work.

Secondly, a series of universal curves based on these parameters ($B_i$ and $l_{g^*}$), including dimensionless temperature profiles at the fin and dimensionless temperature errors, are also presented by graphs.

Starting from the typical data of a particular fin ($k$, $h$ and $c$), which determine the values of the above mentioned parameters using the graph $l_{g^*}=l_{g^*}(B_i)$, the above curves provide interesting information for designing fins.

Finally, universal graphs of efficiency for short fins with adiabatic tip are numerically obtained using the network simulation method. These curves, which can be easily used, are sketched as a function of $L/l_{g^*}$ and using $B_i$ as a parameter.

References:


