Numerical simulation of inviscid compressible flow by higher order numerical schemes

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Abstract: We deal with a numerical simulation of inviscid compressible flow by higher order methods. We present a semi-implicit scheme which is based on the discontinuous Galerkin method for the space discretization and the backward difference formula for the time discretization. Further, we discuss and compare a use of two types of nonreflecting boundary conditions at inflow/outflow parts of boundary. Finally, an example of nonstationary subsonic flow simulation demonstrating an efficiency of the scheme is presented.

Key–Words: Euler equations, discontinuous Galerkin method, backward difference formula, inflow/outflow boundary conditions, Riemann problem

1 Introduction

Our aim is to developed an efficient, robust and accurate numerical scheme for a simulation of unsteady compressible flow. Very promising seems to carry out the space discretization by the discontinuous Galerkin method (DGM), which is based on a piecewise polynomial but discontinuous approximation. It is possible to use DGM also for the time discretization [9] but the most usual approach is the application of the method of lines. In this case, the Runge-Kutta methods are very popular for their simplicity and a high order of accuracy, see [1], [2]. Their drawback is a strong restriction to the choice of the time step. To avoid this disadvantage it is suitable to use an implicit time discretization. A full implicit scheme leads to a necessity to solve a nonlinear system of algebraic equations in each time step which is expensive. Therefore, we proposed in [4] a semi-implicit method, which is based on a suitable linearization of inviscid fluxes. In this paper we generalize this approach to a higher order approximation with respect to the time using the backward difference formulae.

The contents of the rest of the paper is the following. We introduce the considered problem in Section 2 and carried out its discretization with the aid of the higher order semi-implicit DGM in Section 3. We also discuss a choice of the boundary conditions which allow us to solve also flows in an incompressible limit. Section 4 contains an example of unsteady subsonic flow simulation and Section 5 a discussion of the achieved results.

2 Problem formulation

The system of the Euler equations describing 2D inviscid flow can be written in the form

$$\frac{\partial w}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} = 0 \quad \text{in } Q_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain occupied by gas, $T > 0$ is the length of a time interval,

$$w = (w_1, \ldots, w_4)^T = (\rho, \rho v_1, \rho v_2, e)^T$$

is the state vector and

$$f_s(w) = (\rho v_s, \rho v_s v_1 + \delta_{s1} p, \rho v_s v_2 + \delta_{s2} p, (e + p) v_s)^T, \quad s = 1, 2,$$

are the inviscid (Euler) fluxes. We use the following notation: $\rho$ – density, $p$ – pressure, $e$ – total energy, $v = (v_1, v_2)$ – velocity, $\delta_{sk}$ – Kronecker symbol (if $s = k$, then $\delta_{sk} = 1$, else $\delta_{sk} = 0$). The equation of state implies that

$$p = (\gamma - 1) (e - \rho |v|^2 / 2).$$

Here $\gamma > 1$ is the Poisson adiabatic constant.

The system (1) – (4) is hyperbolic. It is equipped with the initial condition

$$w(x, 0) = w^0(x), \quad x \in \Omega,$$

and the boundary conditions

$$B(w) = 0 \quad \text{on } \partial \Omega \times (0, T),$$
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where \( n \) denotes the unit outer normal to \( \partial \Omega \). In order to determine boundary conditions on \( \Gamma_{IO} \), we define the matrix

\[
P(w, n) := \sum_{s=1}^{2} A_s(w)n_s,
\]

where \( n = (n_1, n_2) \in \mathbb{R}^2 \), \( n_1^2 + n_2^2 = 1 \) and

\[
A_s(w) = \frac{\partial f_s(w)}{\partial w}, \quad s = 1, 2,
\]

are the Jacobi matrices of the mappings \( f_s \). Then we prescribe \( m_n \) quantities characterising the state vector \( w \), where \( m_n \) is the number of negative eigenvalues of the matrix \( P(w, n) \) and extrapolate \( m_p \) quantities of \( w \) from interior of \( \Omega \), where \( m_p = 4 - m_n \) is the number of nonnegative eigenvalues of \( P(w, n) \). For details, see, e.g., [5] or [7].

3 Discretization

3.1 Broken Sobolev space

Let \( T_h \) denote a triangulation of the closure \( \overline{\Omega} \) of the domain \( \Omega \) into a finite number of closed elements (triangles or quadrilaterals) \( K \) with mutually disjoint interiors.

We set \( h = \max_{K \in T_h} \text{diam}(K) \). Let \( I \) be a suitable index set such that \( T_h = \{ K_i \}_{i \in I} \). If two elements \( K_i, K_j \in T_h \) contain a nonempty open part of their faces, we put \( \Gamma_{ij} = \Gamma_j \cap \partial K_i \). For \( i \in I \) we set \( s(i) = \{ j \in I; \Gamma_{ij} \text{ exists} \} \). The boundary \( \partial \Omega \) is formed by a finite number of faces of elements \( K_i \) adjacent to \( \partial \Omega \). We denote all these boundary faces by \( S_j \), where \( j \in I_b \) is a suitable index set and put \( \gamma(i) = \{ j \in I_b; S_j \text{ is a face of } K_i \} \). \( \Gamma_{ij} = S_j \) for \( K_i \in T_h \) such that \( S_j \subset \partial K_i \), \( j \in I_b \). Further we define two disjoint subsets \( \gamma_{IO}(i) \) and \( \gamma_{IW}(i) \) corresponding to the boundary parts \( \Gamma_{IO} \) and \( \Gamma_{IW} \), respectively. Obviously, \( \gamma(i) = \gamma_{IO}(i) \cup \gamma_{IW}(i) \). Moreover we put \( S(i) = s(i) \cup \gamma(i) \) and \( n_{ij} = (n_{ij})_1, (n_{ij})_2 \) is the unit outer normal to \( \partial K_i \) on the face \( \Gamma_{ij} \).

Over the triangulation \( T_h \) we define the broken Sobolev space

\[
H^k(\Omega, T_h) = \{ v; v|_K \in H^k(K) \forall K \in T_h \},
\]

where \( H^k(K) = W^{k,2}(K) \) denotes the (classical) Sobolev space on element \( K \). For \( v \in H^1(\Omega, T_h) \) we set

\[
v|_{\Gamma_{ij}} = \text{trace of } v|_{K_i} \text{ on } \Gamma_{ij}, \quad (11)
\]

\[
v|_{\Gamma_{ji}} = \text{trace of } v|_{K_j} \text{ on } \Gamma_{ji},
\]

denoting the traces of \( v \) on \( \Gamma_{ij} = \Gamma_{ji} \), which are different in general.

The approximate solution of problem (1) is sought in the space of discontinuous piecewise polynomial functions \( S_h \), where \( p \) is a positive integer and \( P^p(K) \) denotes the space of all polynomials on \( K \) of degree at most \( p \). Obviously, \( S_h \subset H^1(\Omega, T) \).

3.2 Space semi-discretization

In order to derive the discrete problem, we multiply (1) by a test function \( \varphi \in [H^1(\Omega, T_h)]^4 \), integrate over any element \( K_i, \quad i \in I \), apply Green’s theorem and sum over all \( i \in I \). In this way we obtain the integral identity

\[
\frac{\partial}{\partial t} \sum_{K_i \in T_h} \int_{K_i} w \cdot \varphi \, dx = \int_{\Gamma_{ij}} f_s(w) \cdot \frac{\partial \varphi}{\partial x_s} \, dx \quad (13)
\]

\[
- \sum_{K_i \in T_h} \sum_{j \in S(i)} \int_{\Gamma_{ij}} f_s(w) \cdot \varphi \, (n_{ij})_s \, dS,
\]

which represents a weak form of the Euler equations in the sense of the broken Sobolev space \( H^1(\Omega, T_h) \) defined by (10).

Now we shall introduce the discrete problem approximating identity (13) with the aid of DGM. To evaluate the boundary integrals in (13) we use the approximation

\[
\int_{\Gamma_{ij}} f_s(w(t)) \cdot (n_{ij})_s \cdot \varphi \, dS \quad (14)
\]

\[
\approx \int_{\Gamma_{ij}} H(w(t)|_{\Gamma_{ij}}, w(t)|_{\Gamma_{ji}}, n_{ij}) \cdot \varphi \, dS := \sigma_{ij},
\]

where \( H \) is a numerical flux, \( w(t)|_{\Gamma_{ij}} \) and \( w(t)|_{\Gamma_{ji}} \) are the values of \( w \) on \( \Gamma_{ij} \) considered from the interior and the exterior of \( K_i \), respectively, and at time \( t \). For details, see, e.g. [5] or [7].
It is necessary to specify the meaning of $w(t)\mid_{\Gamma_{ij}}$ for $j \in \gamma(i)$. If $j \in \gamma_W(i)$, then we use the impermeability condition (7) and put in (14) the approximation

$$\sigma_{ij} := \int_{\Gamma_{ij}} F_W(w(t), n_{ij}) \cdot \varphi \, dS, \quad j \in \gamma_W(i)$$

(15)

where $F_W(w, n) \equiv (0, pm_1, pm_2, 0)^T$. The pressure $p$ (given by (4)) is extrapolated on $\Gamma_{ij}$ from $K_i$ and $n = (n_1, n_2) = n_{ij}$. The case $j \in \gamma_{IO}$ is discussed in the Section 3.4.

For $w_h, \varphi_h \in [S_h]^4$ we introduce the forms

$$\langle w_h, \varphi_h \rangle = \int_{\Omega} w_h(x) \cdot \varphi_h(x) \, dx,$$  

(16)

where $w_h(t) : (0, T) \to [S_h]^4, \, \varphi_h \in [S_h]^4, \, t \in (0, T)$.

### 3.3 Full space-time discretization

The problem (17) accompanied by the initial condition (5) exhibits a system of ordinary differential equations for $w_h(t)$ which has to be discretized by a suitable ODE solver. In order to avoid the drawbacks mentioned in Section 1 we follow the approach presented in [4] where a semi-implicit discretization of (17) was presented. We define a suitable linearization of $b_h(\cdot, \cdot)$ by a form

$$b_h(w_h^1, w_h^2, w_h^3), \quad w_h^1, w_h^2, w_h^3 \in [S_h]^4,$$  

(18)

which is linear with respect to its second and third argument and it is consistent with the form $b_h(\cdot, \cdot)$ in the following way

$$b_h(w_h, w_h, \varphi_h) = b_h(w_h, \varphi_h) \quad \forall w_h, \varphi_h \in [S_h]^4.$$  

(19)

This linearization is based on the homogeneity property of the Euler fluxes $f_s$, $s = 1, 2$ and a suitable choice of the numerical flux $H(\cdot, \cdot)$, for detail we refer to the original paper [4].

The main idea of the semi-implicit discretization is to threat the linear part of $b_h$ (represented by its second argument) implicitly and the nonlinear part of $b_h$ (represented by its first argument) explicitly. In order to obtain a sufficiently accurate approximation with respect to the time coordinate we use the so-called backward difference formula (BDF) for the solution ODE problem (17). Moreover, for the nonlinear part of $b_h(\cdot, \cdot)$ we employ a suitable explicit higher order extrapolation which preserve a given order of accuracy and does not destroy the linearity of the problem at each time level.

Let $0 = t_0 < t_1 < \ldots < t_r = T$ be a partition of the interval $(0, T)$ and $\tau_k \equiv t_{k+1} - t_k, \, k = 0, 1, \ldots, r - 1$.

**Definition 1** We define the approximate solution of problem (1) as functions $w_h^k, \, k = 1, \ldots, r$, satisfying the conditions

a) $w_h^{k+1} \in [S_h]^4,$

b) $1 \tau_k \left( \sum_{l=0}^{n} \alpha_l w_h^{k+1-l}, \varphi_h \right) + b_h \left( \sum_{l=1}^{n} \beta_l w_h^{k+1-l}, w_h^{k+1}, \varphi_h \right) = 0$ \quad $\forall \varphi_h \in [S_h]^4, \, k = n - 1, \ldots, r - 1,$

c) $w_h^0$ is $S_h$ approximation of $w^0$,

d) $w_h^l \in [S_h]^4, \, l = 1, \ldots, n - 1$ are given by a suitable one-step method,

where $n \geq 1$ is the degree of the BDF scheme, the coefficients $\alpha_l, \, l = 0, \ldots, n$ and $\beta_l, \, l = 1, \ldots, n$ depend on time steps $\tau_{k-l}, \, l = 0, \ldots, n$.

The relations for the coefficients $\alpha_l, \, l = 0, \ldots, n$ and $\beta_l, \, l = 1, \ldots, n$ were derived in [3] for $n = 1, 2, 3$ and their values for constant time step $\tau_k = \tau, \, k = 1, \ldots, r$ are given in Table 3.3. The problem (20), a) - d) represents a system of linear algebraic equations which is solved by a suitable iterative solver (e.g. GMRES method).

### 3.4 Inflow/outflow boundary conditions

If $\Gamma_{ij} \subset \Gamma_{IO}$, i.e. $j \in \gamma_{IO}(i)$, it is necessary to specify the boundary state $w|_{\Gamma_{ij}}$ appearing in the numerical flux $H$ in the definition of the inviscid form $b_h$. In
virtue of the end of Section 2 it is necessary to prescribe $m_n$ quantities and the others ($m_p = 4 - m_n$) have to be extrapolated. There is $m_n = 3$ for a subsonic inflow and $m_n = 1$ for subsonic outflow. The usual approach is to extrapolate the pressure and prescribe the other quantities at the subsonic inflow and to prescribe the pressure and extrapolated other quantities at the subsonic outflow. Although this approach is satisfactory for transonic flows (see e.g., [7]), its application to low speed flows does not give reasonable results. It is necessary to use nonreflecting boundary conditions transparent for acoustic effects coming from inside of $\Omega$. Therefore, characteristics based boundary conditions are used. Firstly, we remind the approach presented in [6] and then introduce its improvement.

3.4.1 Local linearized Riemann problem

In [6] we employed a solution of the local linearized Riemann problem considered on each boundary edge. Let $\Gamma_{ij} \subset \Gamma_{10}$, $w_{ij} = w|_{\Gamma_{ij}}$ and $w_{BC}$ be a prescribed boundary state at the inflow or outflow. Using the rotational invariance, we transform the Euler equations (1) to the coordinates $\tilde{x}_1$, parallel with the normal direction $n$ to $\Gamma_{ij}$, and $\tilde{x}_2$, tangential, neglect the derivative with respect to $\tilde{x}_2$ and linearize the system around the state $q_{ij} = Q(n_{ij})w|_{\Gamma_{ij}}$, where $Q(n_{ij})$ is the rotational matrix. Then we obtain the linear system

$$ \frac{\partial q}{\partial t} + A_1(q_{ij}) \frac{\partial q}{\partial \tilde{x}_1} = 0, \quad (21) $$

for the transformed vector-valued function $q = Q(n_{ij})w$, considered in the set $(-\infty, \infty) \times (0, \infty)$ and equipped with the initial condition

$$ q(\tilde{x}_1, 0) = q_{ij}, \quad \tilde{x}_1 < 0, \quad (22) $$
$$ q(\tilde{x}_1, 0) = q_{BC}, \quad \tilde{x}_1 > 0, $$

where $q_{BC} = Q(n_{ij})w_{BC}$.

The solution of the linear problem (21) – (22) can be easily found by computing eigenvalues and eigenvectors of matrix $A_1(q_{ij})$ (for explicit formulae see, e.g., [7], Section 3.1). The sought boundary state $w|_{\Gamma_{ij}}$ is defined as (according [6])

$$ w|_{\Gamma_{ij}} = Q^{-1}(n_{ij})q(\tilde{x}_1 = 0), \quad (23) $$

where $q(\tilde{x}_1 = 0)$ is the solution of (21) – (22) at $\tilde{x}_1 = 0$ and $t > 0$.

Although the results from [6] (using the definition of the boundary state (23)) look very nice even for very small Mach numbers, we observe (having a different computer implementation) a small instabilities of this type of boundary conditions. Figure 1, left, shows this type of instabilities for a simulation of an inviscid flow around a unit circle with far-field Mach number $M_\infty = 0.1$ using piecewise cubic approximation (i.e., $p = 3$ in (12)). We suppose that the presence of the instabilities is caused by a different choice of the basis functions in comparison with the mentioned references. Therefore, we propose a modification of this approach, which is based on the solution of the local exact Riemann problem and does not suffer from this drawback.

3.4.2 Local exact Riemann problem

We use the similar strategy as in Section 3.4.1 but avoid the linearization around the state $q_{ij}$. Therefore, instead of system (21) we consider the following
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nonlinear problem

\[ \frac{\partial q}{\partial t} + A_1(q) \frac{\partial q}{\partial x_1} = 0, \]

(24)
in \((-\infty, \infty) \times (0, \infty)\) and with the initial condition (22). The boundary state \( w|_{\Gamma_{j1}} \) is defined by

\[ w|_{\Gamma_{j1}} = Q^{-1}(n_{ij})^T \dot{q}(\tilde{x}_1 = 0), \]

(25)

where \( \dot{q}(\tilde{x}_1 = 0) \) is the solution of (24) with (22) at \( \tilde{x}_1 = 0 \) and \( t > 0 \).

The solution \( \dot{q}(\tilde{x}_1 = 0) \) is given by a set of nonlinear algebraic equations which can be solved numerically by a suitable iterative process, for more details see [8]. Using this approach, we have not observed any instabilities as in the previous one, see Figure 1, right. Numerical experiments (not presented in this paper) give very nice solutions even for \( M_\infty = 10^{-3} \).

4 Numerical example

We present an example of unsteady subsonic flow through the GAMM channel, see Figure 2, left. As an initial condition we take a steady state solution for the inflow Mach number \( M = 0.5 \) (see Figure 2, right showing the isolines of pressure), which was obtained using the boundary condition described in Section 3.4.2 with \( w_{BC} = (1, 1, 0, 7.642857) \) at the inflow and outflow parts of the channel, which, using (4) gives the inflow/output pressure \( p_0 = 2.857142857 \).

At \( t = 0 \) we start to periodically modify the pressure at outflow of the channel according the formulae

\[ p = p_0(1 + 0.1 \sin(\pi t)) \]  

(26)

and consequently (using (4)) the value of \( w_{BC} \) at the outflow. Figure 3 shows the isolines of pressure at several time instants. These results were achieved on relatively coarse triangular grid having 598 triangles (Figure 2, left) with piecewise cubic polynomial approximation (i.e., \( p = 3 \) in (12)) and the third order time discretization (i.e., \( n = 3 \) in (20)). We observe a nice periodical propagation of pressure waves from the right to the left of the channel and also any reflection of these waves from the inflow which verifies the correctness of the presented boundary conditions.

5 Conclusion

We presented a space-time higher order scheme for the numerical simulation of unsteady compressible flow with an application for a subsonic flow through the GAMM channel. The nonreflecting boundary conditions and the suitable linearization with the extrapolation for the time discretization gives the efficient tools for computational fluid dynamics. It would be also interesting to carry out a simulation of transonic flow where some discontinuities (shock waves) appear. A correct numerical capturing of these singularities requires a special stabilization technique which is a subject of a further research.

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References:


Figure 2: GAMM channel, used triangulation (left) and steady-state solution – isolines of pressure (right)

Figure 3: GAMM channel with oscillating outflow pressure, isolines of pressure at several time instants