An $L_{\omega_1\omega_1}$ Axiomatization of the Linear Archimedean Continua as Merely Relational Structures

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Abstract. – We have chosen the language $L_{\omega_1\omega_1}$ in which to formulate the axioms of two systems of the linear Archimedean continua – the point-based system, $S_p$, and the stretch-based system, $S_I$ – for the following reasons: 1. It enables us to formulate all the axioms of each system in one and the same language; 2. It makes it possible to apply, without any modification, Arsenijević's two sets of rules for translating formulas of each of these systems into formulas of the other, in spite of the fact that these rules were originally formulated in a first-order language for systems that are not continuous but dense only; 3. It enables us to speak about an infinite number of elements of a continuous structure by mentioning explicitly only denumerably many of them; 4. In this way we can formulate not only Cantor's coherence condition for linear continuity but also express the large-scale and small-scale variants of the Archimedean axiom without any reference, either explicit or implicit, to a metric; 5. The models of the two axiom systems are structures that need not be relational-operational but only relational, which means that we can speak of the linear geometric continua directly and not only via the field of real numbers (numbers will occur as subscripts only, and they will be limited to the natural numbers).

Key-Words: Linear continuum, $L_{\omega_1\omega_1}$, point-based, stretch-based axiomatization, trivial difference, Archimedean axiom

1 Introduction
Cantor established the point-based conception of the continuum, stating that a linearly ordered set of null-dimensional points actually makes up a continuum if the set is perfect and coherent (zusammenhängend) ([7], p. 194). But though the majority of mathematicians and philosophers sided with Cantor’s view (cf. [11]), in the last four decades a number of authors revived the Aristotelian stretch-based approach (see [1], [3]-[6], [8], [10], [12]-[14], [16]-[19]). However, in spite of the fact that after any axiomatization of each of the two systems – let us call them $S_p$ and $S_I$, respectively – there will be no model in which the variables of $S_p$ and the variables of $S_I$ range over elements of one and the same basic set, there is a strong intuitive similarity and a possible “systematic connection” between the two systems ([3], p. 84, cf. also [5]) that suggests that they should be classified as only trivially different. The underlying idea is that stretches can be introduced into $S_p$ as intervals between two points while points can be introduced into $S_I$ as abutment places of two stretches (or two equivalence classes of stretches). The fact that stretches are originally neither closed nor open can be compensated by letting them stand for the closed intervals in contrast to sets of an infinite number of stretches having either greatest lower or least upper bounds or both, which represent half-open and open intervals, respectively.

2 Problem Formulation
In [2], Arsenijević defined the generalized concepts of trivial syntactical and semantic differences between two formal theories and
showed, by using two mutually non-inverse sets of translation rules, that two axiomatic systems implicitly defining point structures and stretch structures that are dense are just trivially different in the defined sense. Now, we want to show that this result holds also when the systems are extended so as to satisfy Cantor's second condition, i.e., if the structures are not only dense but also continuous. The main problem in showing this consists in the fact that Arsenijević’s rules are tailored to first-order languages, whereas the continuity axiom is normally formulated in a second-order language. We shall solve this problem by choosing the language $\mathcal{L}_{01}$ in which to formulate the axioms of two systems, which allows the application of Arsenijević’s rules without any modification. At the same time, we shall both avoid some unnecessary commitments of the second-order language and always mention only a denumerable number of elements of the continuum.

Another problem is that the two resulting systems of the linear continuum in which numbers are neither mentioned nor used (except as variable subscripts) are insensitive to a distinction between Archimedean and non-Archimedean structures, which both belong to the class of their models (cf. [9]). Since there is no metric, obtainable either geometrically via the equality relation holding between stretches or arithmetically through the operations of multiplication and division, the large-scale and the small-scale variant of the Archimedean axiom must be formulated purely topologically by mentioning denumerably many of points and stretches only. This constitutes an important novelty of our approach.

3 Comparison between $S_P$ and $S_I$

3.1 Axiomatization of the Point-Based System

Let, in the intended model of $S_P$, the individual variables $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n, \ldots$ range over a set of null-dimensional points, and let the relation symbols $\equiv$, $<$, and $>$ be interpreted as the identity, precedence, and succession relations respectively. Let the elementary wffs of $S_P$ be

$\alpha_m \equiv \alpha_n$, $\alpha_m < \alpha_n$, and $\alpha_m > \alpha_n$, where $\alpha_m > \alpha_n \iff \text{def. } \alpha_n < \alpha_m$. Finally, let the axiom schemes of $S_P$ be the following twelve formulas, which we shall refer to as $(A_P1)$, $(A_P2), \ldots, (A_P12)$:

1. $(\alpha_m) \rightarrow \alpha_m < \alpha_n$
2. $(\alpha_i)(\alpha_m)(\alpha_n)(\alpha_i < \alpha_m \land \alpha_m < \alpha_n \Rightarrow \alpha_i < \alpha_n)$
3. $(\alpha_i)(\alpha_m)(\alpha_n)(\alpha_i < \alpha_m \land \alpha_i < \alpha_m \land \alpha_m = \alpha_i \Rightarrow \alpha_i = \alpha_m)$
4. $(\alpha_i)(\alpha_m)(\alpha_n)(\alpha_i \equiv \alpha_m \land \alpha_i < \alpha_n \Rightarrow \alpha_m < \alpha_n)$
5. $(\alpha_i)(\alpha_m)(\alpha_n)(\alpha_i \equiv \alpha_m \land \alpha_m < \alpha_i \Rightarrow \alpha_i < \alpha_m)$
6. $(\alpha_m)(\exists \alpha_n) \alpha_m < \alpha_n$
7. $(\alpha_m)(\exists \alpha_n) \alpha_m < \alpha_n$
8. $(\alpha_m)(\alpha_n)(\alpha_m < \alpha_n \Rightarrow (\exists \alpha_n)(\alpha_m < \alpha_l \land \alpha_l < \alpha_n))$
9. $(\alpha_1)(\alpha_2) \ldots (\alpha_i) \ldots (\exists \beta_1)(\land_{1 \leq i < n} \alpha_i < \beta_1) \Rightarrow$
   $(\exists \gamma_1)(\land_{1 \leq i < n} \alpha_i < \gamma_1 \land$
   $\neg(\exists \delta_1)(\land_{1 \leq i < n} \alpha_i < \delta_1 \land \delta_1 < \gamma_1))$
10. $(\alpha_1)(\alpha_2) \ldots (\alpha_i) \ldots (\exists \beta_1)(\land_{1 \leq i < n} \alpha_i > \beta_1) \Rightarrow$
   $(\exists \gamma_1)(\land_{1 \leq i < n} \alpha_i > \gamma_1 \land$
   $\neg(\exists \delta_1)(\land_{1 \leq i < n} \alpha_i > \delta_1 \land \delta_1 < \gamma_1))$
11. $(\exists \alpha_1)(\exists \alpha_2) \ldots (\exists \alpha_i) \ldots (\exists \alpha_n) \land \alpha_i < \alpha_1 \land$
   $\land_{1 \leq i < n} \alpha_{2i-1} < \alpha_{2i+1} \land \land_{1 \leq i < n} \alpha_{2i+2} < \alpha_{2i} \land$
   $\land \beta \land_{1 \leq i < n} (\alpha_i < \beta \land \beta < \alpha_{i+2} \Rightarrow$
   $\land_{1 \leq i < n} \neg(\exists \alpha_k)(\land_{1 \leq j < \alpha_k}(\alpha_i < \gamma \land \gamma < \alpha_j))$

3.2 Axiomatization of the Stretch-Based System

Let, in the intended model of $S_I$, the individual variables $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n, \ldots$ range over one-dimensional stretches, and let the relation symbols $\equiv$, $<$, and $>$ be interpreted as the identity, precedence, succession, abutment, overlapping, and inclusion relations respectively. Let the elementary wffs be $a_m = a_n$, $a_m < a_n$, $a_m > a_n$, $a_m \nvdash a_n$, $a_m \overset{1}{\nvdash} a_n$, $a_m \overset{1}{\nvdash} a_n$, $a_m \cap a_n$, and $a_m \subset a_n$, where $a_m > a_n \iff \text{def. } a_n < a_m$ and $a_m \nvdash a_n \iff \text{def. } a_m \nvdash a_n$, $a_m \overset{1}{\nvdash} a_n \iff \text{def. } a_m \nvdash a_n \land \neg(\exists a_i)(a_m < a_i \land a_i < a_n)$, $a_m \cap a_n \iff \text{def. } (\exists a_i)(\exists a_k)(a_i < a_n \land a_i < a_m \land a_i < a_k \land a_k > a_n)$, $a_m \subset a_n \iff \text{def. } a_m = a_n \land (a_i)(a_i \cap a_m \Rightarrow a_i \cap a_n)$. 
Finally, let axiom schemes of $S_l$ be the following twelve formulas, which we shall refer to as $(A_1), (A_2), \ldots, (A_{12})$:

1. $a_n \prec a_n < a_n$

2. $(a_k(a_1(a_2(a_3(\ldots(a_4(a_m a_n \prec a_m \land a_t \prec a_m \Rightarrow a_k \prec a_n \prec a_t \land a_j \prec a_m))\ldots))))$

3. $(a_k(a_i a_n \prec a_m \Rightarrow a_m a_k a_i a_t \prec a_m))$

4. $(a_k(a_i a_m a_n a_k a_i a_t a_m a_k a_t \land a_m a_t \Rightarrow a_k \prec a_n))$

5. $(a_k(a_m a_n a_k a_i a_t a_m a_k a_t a_m a_n \Rightarrow a_k \prec a_n))$

6. $(a_m(\exists a_n) a_m \prec a_n)$

7. $(a_m(\exists a_n) a_n \prec a_m)$

8. $(a_m(\exists a_n) a_n \prec a_m)$

9. $(a_i(a_2(a_3(\ldots(a_k(\ldots(\exists u)(\land_{1 \leq i < o} a_i \prec u) \Rightarrow$

\begin{align*}
& (\exists v)(\land_{1 \leq i < o} a_i \prec v \land \neg (\exists w) (\land_{1 \leq i < o} a_i \prec w \land w < v)))
\end{align*}

10. $(a_i(a_2(a_3(\ldots(a_k(\ldots(\exists u)(\land_{1 \leq i < o} a_i \prec u) \Rightarrow$

\begin{align*}
& (\exists v)(\land_{1 \leq i < o} a_i \prec v \land \neg (\exists w) (\land_{1 \leq i < o} a_i \prec w \land w < v)))
\end{align*}

11. $(\exists a_1(\exists a_2)(\ldots(\exists a_n)\ldots)\ldots(a_2 a_1 \land_{1 \leq i < o} a_i \prec a_{2i+1} \land_{1 \leq i < o} a_{2i} a_{2i+1} \land (b)\land_{1 \leq i < o} a_i \prec b \land b \prec a_i))$

12. $(\exists a_1(\exists a_2)(\ldots(\exists a_n)\ldots)(b)\land_{1 \leq i < o} a_i \prec a_i \Rightarrow$

\begin{align*}
& (\land_{1 \leq i < o} b \prec a_i \land \land_{1 \leq i < o} b \land (b)) \land
\end{align*}

\begin{align*}
& (c)\land_{1 \leq i < o} a_i \prec a_i \Rightarrow \land_{1 \leq i < o} a_i c \land
\end{align*}

\begin{align*}
& (d)\land_{1 \leq i < o} (a_i \prec d \land d \prec a_i) \land \\
& (e)\land_{1 \leq i < o} (a_i \prec e \land e \prec a_i))
\end{align*}

3.3 Comments on some Axioms

The interpretation of the first eight axioms of both systems needs no special comments. They implicitly define dense, undefined, and linearly ordered structures. However, the rest of the axioms need some comments.

$Ad (A_{P9})$ and $(A_{P10})$, and $(A_{S9})$ and $(A_{S10})$. According to Cantor’s definition, a linearly ordered set of null-dimensional points is “perfekt” (i.e., dense) if each element of the set is an accumulation point of an infinite number of elements of the set, whereas it is “zusammenhängend” (i.e., coherent) if each accumulation point of an infinite number of elements of the set is also an element of the set itself ([7], p. 194). Now, while the first condition is met by axiom $(A_{P9})$, the second is met, for the whole class of isomorphic models, only by two axioms, $(A_{P9})$ and $(A_{P10})$, which state the existence of the least upper and the greatest lower bound, respectively. It might be of interest to note why it is so. Namely, we need both $(A_{P9})$ and $(A_{P10})$ in order to make the class of all the models for $S_P$ isomorphic. Let us suppose that, though the elements of the intended model of $S_P$ are points, they are, instead (as in [8]), the sets of numbers of closed intervals between any two numbers $a$ and $b$ such that $a \in Q$ and $b \in R$, and $< \approx$ is interpreted as “is a proper subset of”. Then, the relational structure $\langle \{a, b\} | a \in Q, b \in R \rangle$ satisfies the set of axioms $(A_{P1}), \ldots, (A_{P9})$ but the coherence condition is not met. Let us replace, for instance, the set of intervals $[a_1, b], [a_2, b], \ldots, [a_n, b] \ldots$ such that $a_1$ is a number smaller than $b$ and any $a_{n+1}$ is smaller than $a_n$, and where $\pi$ is the accumulation point of the set of numbers $a_1, a_2, \ldots, a_n, \ldots$. There is no greatest lower bound for this set of intervals, in spite of the fact that the least upper bound always exists. – A similar example can be constructed for showing that we need both $(A_{S9})$ and $(A_{S10})$.

$Ad (A_{P11})$ and $(A_{P11})$. The intended meaning of the large-scale variant of the Archimedean axiom can be expressed by choosing a denumerable set of discrete points (in $S_P$) or abutting stretches (in $S_l$) distributed over the whole continuum and claiming that for any element of the structure there are two distinct elements (points or stretches) from the given sets such that one of them lies on one side and the other on the other side of the given element (point or stretch). As a consequence, a theorem (whose stretch-based version will be proved below) stating the compactness property of the corresponding model exhibits the intended meaning of the Archimedean axiom in its most obvious form.

$Ad (A_{P12})$ and $(A_{P12})$. For precluding infinitesimals in $S_P$, we have to claim that it is possible to choose a denumerable set of dense points that covers the continuum in such a way that for any two points there is a point from the chosen set that lies between them. In $S_l$, we have to claim that there are no stretches, like monads.
in the Robinsonian non-standard field \(*R\) (cf. [15], p. 57), which are impenetrable, from both sides, by some two members of a chosen denumerable set of abutting and dense stretches.

3.4 The Triviality of the Difference between \(S_P\) and \(S_I\)

In order to show that the two axiom systems, \(S_P\) and \(S_I\), are only trivially different in the sense defined in [2], we shall first cite two sets of translation rules.

Let \(f\) be a function \(f : \alpha_n \longrightarrow \langle a_{2n-1}, a_{2n} \rangle (n = 1, 2, \ldots)\) mapping variables of \(S_P\) into ordered pairs of variables of \(S_P\), and let \(C_1\)-\(C_5\) be the following translation rules providing a 1-1 translation of all the wffs of \(S_P\) into \(S_I\) (where \(=^C\) is to be understood analogously to \(\equiv^C\)):

\[ \begin{align*}
C_1^* : & a_n = a_m =^C a_{2n-1} < a_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}, \\
C_2^* : & a_n < a_m =^C a_{2n-1} < a_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \neg \alpha_{2m-1} < a_{2n}, \\
C_3^* : & \neg F_I =^C \neg \Phi^C(F_I), \text{ where } F_I \text{ is a wff of } S_I \text{ translated according to } C_1^* - C_5^* \text{ into wff } C(F_I) \text{ of } S_P, \\
C_4^* : & F_I \wedge F'' =^C \Phi^C(F_I)^C \wedge \Phi^C(F''), \text{ where } \wedge \text{ stands for } \Rightarrow \text{ or } \wedge \text{ or } \vee \text{ or } \Leftrightarrow, \text{ and } F_I \text{ and } F'' \text{ stands for two wffs of } S_P \text{ translated according to } C_1^* - C_5^* \text{ into two wffs of } S_I, C(F_I)^C \text{ and } C(F''^C) \text{ respectively}, \\
C_5^* : & (a_n) \Omega^C(\alpha_n) =^C(c_{2n-1}(a_{2n}))((a_{2n-1} < a_{2n}) \Rightarrow \Omega^C(a_{2n}), \text{ and} \\
& (\exists a_n) \Omega^C(\alpha_n) =^C(\exists a_{2n-1})((a_{2n-1} < a_{2n}) \wedge \Omega^C(a_{2n-1}, a_{2n}), \\
& \text{ where } \Omega^C(\alpha_n) \text{ is a formula of } S_P \text{ translated into formula } \Omega^C(a_{2n-1}, a_{2n}) \text{ of } S_P \text{ according to } C_1^* - C_5^*.
\end{align*} \]

Let \(f^*\) be a function \(f^* : a_n \longrightarrow \langle a_{2n-1}, a_{2n} \rangle (n = 1, 2, \ldots)\) mapping variables of \(S_I\) into ordered pairs of variables of \(S_P\), and let \(C_1^* - C_5^*\) be the following translation rules providing a 1-1 translation of all the wffs of \(S_I\) into a subset of the wffs of \(S_P\) (where \(=^C\) is to be understood analogously to \(\equiv^C\)):

\[ \begin{align*}
C_1 : & a_n = a_m = a_{2n-1} < a_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \alpha_{2n-1} \equiv \alpha_{2m-1} \wedge \alpha_{2n} \equiv \alpha_{2m}, \\
C_2 : & a_n < a_m = a_{2n-1} < a_{2n} \wedge \alpha_{2m-1} < \alpha_{2m} \wedge \neg \alpha_{2m-1} < a_{2n}, \\
C_3 : & \neg F_I =^C \neg \Phi^C(F_I), \text{ where } F_I \text{ is a wff of } S_I \text{ translated according to } C_1^* - C_5^* \text{ into wff } C(F_I) \text{ of } S_P, \\
C_4 : & F_I \lor F'' =^C \Phi^C(F_I)^C \lor \Phi^C(F''^C), \text{ where } \lor \text{ stands for } \Rightarrow \text{ or } \wedge \text{ or } \vee \text{ or } \Leftrightarrow, \text{ and } F_I \text{ and } F'' \text{ stands for two wffs of } S_P \text{ translated according to } C_1^* - C_5^* \text{ into two wffs of } S_P, C(F_I)^C \text{ and } C(F''^C) \text{ respectively}, \\
C_5 : & (a_n) \Omega(\alpha_n) =^C(c_{2n-1}(a_{2n}))((a_{2n-1} < a_{2n}) \Rightarrow \Phi^C(a_{2n-1}, a_{2n}), \\
& (\exists a_n) \Omega(\alpha_n) =^C(\exists a_{2n-1})((a_{2n-1} < a_{2n}) \wedge \Phi^C(a_{2n-1}, a_{2n}), \\
& \text{ where } \Omega(\alpha_n) \text{ is a formula of } S_I \text{ translated into formula } \Phi^C(a_{2n-1}, a_{2n}) \text{ of } S_P \text{ according to } C_1^* - C_5^*.
\end{align*} \]

In [2], Arsenijević has shown that by using \(C_1^* - C_5^*\) for translating \((A_1), \ldots, (A_8)\) into \(S_I\) and \((A_1), \ldots, (A_8)\) into \(S_P\), respectively, we always get theorems. Now, the same holds for the translations of \((A_9), \ldots, (A_{12})\) into \(S_I\), \((A_9), \ldots, (A_{12})\) into \(S_P\). Let us prove within \(S_P\) the translation of \((A_9)^*\), which will be (after an appropriate shortening of the resulting formula) denoted by \((A_9)^*\).

\[ \begin{align*}
(A_9)^* : & (a_1)(a_2) \ldots (a_i) (\wedge_{1 \leq i \leq 9} a_{2i-1} < a_{2i}) \Rightarrow \\
& (\exists b_1)(\exists b_2)(b_1 < b_2 \wedge (\wedge_{1 \leq i \leq 9} a_i < b_2)) \Rightarrow \\
& (\exists c_1)(\exists c_2)(c_1 < c_2 \wedge (\wedge_{1 \leq i \leq 9} a_i < c_2) \wedge \\
& \neg(\exists d_1)(\exists d_2)(d_1 < d_2 \wedge ((\wedge_{1 \leq i \leq 9} a_i < d_2) \wedge \\
& \neg d_1 < c_2 \wedge \neg d_1 < c_2))).
\end{align*} \]

**Proof for \((A_9)^*\)**

Let us assume both \(\wedge_{1 \leq i \leq 9} a_{2i-1} < a_{2i}\)
and 
\[(3b_1)(\exists b_2)(b_1 < b_2 \land (\land_{1 \leq i < \omega} a_i < b_2)),\]
which are the two antecedents of \((A_9)\)\(^*\). Now, since for any \(i \ (1 \leq i < \omega)\), \(a_i < b_2\), it follows
directly from \((A_9)\) that there is \(v\) such that \(a_i < v\) and, for no \(w\), both \(a_i < w\) and \(w < v\).

Let us now assume, contrary to the statement of the consequent of \((A_9)\), that for any two \(c_1, c_2\) such that \(c_1 < c_2\) and for any \(i \ (1 \leq i < \omega)\) \(a_i < c_2\), there are always \(d_1\) and \(d_2\) such that \(d_1 < d_2\) and for any \(i \ (1 \leq i < \omega)\) \(a_i < d_2\), so that \(d_1 < c_2\) and \(d_1 < c_2\). But then, if we take \(c_2\) to be just \(v\) from the consequent of \((A_9)\) (and \(c_1\) any interval such that \(c_1 < c_2\), the assumption that for any \(i \ (1 \leq i < \omega)\) \(a_i < c_2\) but \(d_1 < c_2\) and \(d_1 < c_2\) contradicts the choice of \(c_2\), since \(c_2\) = \(v\), then, according to \((A_9)\), for any \(d_1\) and \(d_2\) such that \(d_1 < d_2\) and for any \(i \ (1 \leq i < \omega)\) \(a_i < d_2\), it cannot be that \(d_1 < c_2\) and \(d_1 < c_2\). (Q.E.D.)

4 Application
Let us, finally, prove two theorems in \(S_I\) that are of interest for different reasons. The first of them makes clear what is the trick of our formulation of the large-scale variant of the Archimedean axiom via a chosen denumerable set of abutting stretches distributed over the both sides of the continuum: it is sufficient to have effective control over the continuum by a denumerable number of its discrete elements for making any of its elements surpassable in a finite number of steps, which means that the essence of the Archimedean axiom is topological, having nothing to do with a presupposed metric and depending on no arithmetical operation. The second theorem is a variant of Bolzano-Weierstrass’ statement, which turns out to be not only a consequence of the small-scale variant of the Archimedean axiom but also not to be provable without it.

The \(S_I\) formulation of the Theorem stating the compactness property for stretches:
\((c)(d)(c < d)\)
5 Conclusion
After formulating in \( L_{0}(L) \) the axioms of the Cantorian and the Aristotelian systems of the linear Archimedean continuum, we have shown how, by using appropriate translation rules, the axiom of the point-based system \((A_{P}9)\), which states the existence of the lowest upper bound, can be proved as a theorem in the stretch-based system. In a similar way, it can be shown that after translating \((A_{P}10), (A_{P}11),\) and \((A_{P}12)\) into \( S_{I}\), and \((A_{I}9), (A_{I}10), (A_{I}11)\), and \((A_{I}12)\) into \( S_{R}\), we also get theorems of \( S_{I}\) and \( S_{R}\), respectively. This means that \( S_{P}\) and \( S_{I}\) are only trivially different according to Arsenijević's definition given in [2]. In section 4, we have proved, by using the stretch-based system, two important theorems of classical arithmetic. These proofs strongly suggest that other classical theorems concerning the linear Archimedean continuum can also be formulated as being about merely relational structures and proved on the basis of the cited axioms without the use of the algebraic relational-operational structure of real numbers, which presents a prospect for further investigations.

References


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