Fuzzy chromatic number and fuzzy defining number of certain fuzzy graphs

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Abstract: In this paper we introduce the concepts of fuzzy defining number in some fuzzy graphs. We determine the chromatic number for several classes of fuzzy graphs and obtain bounds for them. We also determine the fuzzy defining number of the classes.

Key Words: fuzzy graph, fuzzy chromatic number, union, join, the Cartesian product.

1 Introduction

The motivation for introducing fuzzy graphs is reflected upon by Rosenfeld in (Zadeh, 1975). Then the fuzzy graphs were discussed in (Kaufman, 1977, Dubois, Prade, 1980, Volkmann, 1991) and in some others papers and books. There are many problems, which can be solved with the help of the fuzzy graphs. Graph theory has proved to be an extremely useful tool for solving many tasks (Berge, 1989), [2]. Let \( V \) be a finite nonempty set. Let \( E \) be the collection of all two-element subsets of \( V \) (\( E \subseteq V \times V \)). A fuzzy graph \( \tilde{G} = (\sigma, \mu) \) is a set with two functions \( \sigma : V \rightarrow I \) and \( \mu : E \rightarrow I' \) such that \( \mu(\{x, y\}) \leq \sigma(x) \wedge \sigma(y) \) for all \( x, y \in V \). Where \( \sigma : V \rightarrow I \) and \( \mu : E \rightarrow I' \) are the vertex membership function on the \( I \) and edge membership function on the \( I' \) respectively, and \( \sigma(x), \mu(\{x, y\}) \) reflects the ambiguity of the assertion \( x \) belongs to \( I \), \( I' \) respectively, [2]. In classical fuzzy-set theory the set \( I \) is usually defined as the closed interval [0,1], in such a way that \( \sigma(x) = 0 \) indicates that \( x \) does not belong to \( I \), \( \sigma(x) = 1 \) indicates that \( x \) strictly belongs to \( I \), and any intermediate value represents the degree in which \( x \) could belong to \( I \). However, the set \( I \) could be a discrete set of the form \( I = \{0, 1, ..., k\} \), where \( \sigma(x) \leq \sigma(x') \) indicates that the degree of membership function \( x \) to \( I \) is lower than the degree of membership function \( x' \). In general, the set \( I \) can be any ordered set, not necessarily numerical, for instance, \( I = \{n, l, m, h, t, c, p\} \), where \( n, l, m, h, t, c \) and \( p \) denote the fuzzy degrees null, low, medium, high, total, complete and perfect, respectively, [8].

Graph theory has numerous applications to problems in systems analysis, operations research, mobile telephone service, missile guidance, transportation, and economics. In many cases, however, some aspects of the graph-theoretic problem are uncertain. In these cases, it can be useful to deal with this uncertainty using the methods of fuzzy logic. Graph coloring is one of the most studied problems of combinatorial optimization. An important area of application of the coloring problem is management science,[3, 5, 9, 12, 13]..

1.1 Preliminaries

Definition 1 Let \( S \) be the available color set. Let \( d \) be the dissimilarity measure defined by \( d : S \times S \rightarrow \mathbb{R}_+ \) with the following properties:

1. \( d(r, s) \geq 0 \) \quad \forall r, s \in S,
2. \( d(r, s) = 0 \iff r = s \) \quad \forall r, s \in S,
3. \( d(r, s) = d(s, r) \) \quad \forall r, s \in S.

This dissimilarity measure \( d \) can take into account the incompatibility degree in the sense that the more incompatible two vertices are, the more distant their associated colors. In this way, an extended coloring function is introduced [8].

Definition 2 Let \( I' \) be a domain of the edge membership function of the fuzzy graph \( \tilde{G} = (\sigma, \mu) \), we defined \( f : I' \rightarrow [0, \infty) \) be a non-negative, non-decreasing (with respect to the order \( \leq \)) and real scale function, i.e.
Given $I$, a be domain of the vertex membership function of the fuzzy graph $G = (\sigma, \mu)$, we defined $g : I \times I \rightarrow [0, \infty)$ be a non-negative, non-decreasing (with respect to the order $\leq$) real scale function, i.e.,

$$\forall \sigma_i, \sigma_j, \sigma_i', \sigma_j' \in I \text{ such that } \mu_{ij} \leq \mu_{i'j'} \Rightarrow g(\sigma_i, \sigma_j) \leq g(\sigma_i', \sigma_j') \quad (3)$$

**Definition 6** Given a fuzzy graph $G = (\sigma, \mu)$, a color set $S$, a dissimilarity measure $d$ defined on $S$ and scale functions $f, g$, a $(d, f, g)$-extended coloring function of $G$, denoted as $C_{d,f,g}$, for short as $C$, is a mapping:

$$C : V \rightarrow S$$

with the property:

$$d(C(i), C(j)) \geq f(\mu_{ij}) \forall i, j \in V, \text{ such that } i \neq j.$$  

$$d(C(i), C(j)) \geq g(\sigma_i, \sigma_j) \forall i, j \in V, \text{ such that } i \neq j.$$  

(4)

A $(d, f, g)$-extended $k$-coloring $C^k_{d,f,g}$, for short $C^k$, is a $(d, f, g)$-extended coloring function with no more than $k$ different colors: $C^k : V \rightarrow S$, where $S = \{1, \ldots, k\}$.

**Definition 7** Given a fuzzy graph $G$, a dissimilarity measure $d$ and scale functions $f, g$, the minimum value $k$ for which a $(d, f, g)$-extended $k$-coloring of $G$ exists is called the $(d, f, g)$-chromatic number of $G$ and it is denoted by $\chi_{d,f,g}(G)$ [8].

**Definition 8** Given a fuzzy graph $G = (\sigma, \mu)$, a set of vertices $S$ with an assignment of colors to them is called a defining set of the vertex coloring of $G$ if there exists a unique extension of the colors of $S$ to a $\chi_{d,f,g}(G)$-coloring of the vertices of $G$. A defining set with minimum cardinality, is called a minimum defining set and its cardinality, the defining number, is denoted by $d_{d,f,g}(G, \chi_{d,f,g})$, for short $d_{d,f,g}$ [7].

2 Some results

**Definition 9** Let $\sigma : V \rightarrow I$ be a fuzzy subset of $V$. Then the complete fuzzy graph on $\sigma$ is defined to be $(\sigma, \mu)$ where $\mu(xy) = \sigma(x) \land \sigma(y)$ for all $xy \in E$ and is denoted by $K_\sigma$ [11].

**Proposition 8** Let $K_\sigma$ be a complete fuzzy graph, and $d$ be a dissimilarity measure and $f, g$ be the scale functions where $g(\sigma(i), \sigma(j)) = f(\mu_{ij})$ for all $i, j \in V$. Then $1 \leq \chi_{d,f,g}(K_\sigma) \leq 1 + (n - 1)D$ where $D = \max\{f(\mu(xy)) | xy \in E\}$ and $n = |V|$.

**Proof:** By definition any edge of $K_\sigma$ is an effective edge. Consider the greedy coloring, if for $v_i \in v_j$, $D = f(\mu(v_i v_j))$ then by rearrangement in number of vertices we put $D = f(\mu(v_1 v_2))$. Now we can color the first vertex by number 1, the second vertex we can color by number 1 + $D$ and then third vertex is colored by $1 + 2D$ since $D \geq f(\mu(3y)) \forall y \in V$. Also the forth vertex is colored with the number 1 + 3$D$ because $D \geq f(\mu(3y))$. In this way we found the $n_{th}$ vertex color by $1 + (n - 1)D$. This coloring is an upper bound for coloring of this graph, hence the $D$ is a maximum value scale function for every edge.

**Definition 10** A path $P$ a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices $(u_0, u_1, u_2, \ldots, u_n)$ such that $\mu(u_i, u_{i+1}) > 0$, $1 \leq i \leq n$ and $n$ is called the length of $P$. The path $P$ is called $u_0 - u_n$ path [11].

**Proposition 10** Let $P$ be a fuzzy path, a $d$ be dissimilarity measure and $f, g$ be the scale functions where $g(\sigma(i), \sigma(j)) = f(\mu_{ij})$ for all $i, j \in V$. Then $1 \leq \chi_{d,f,g}(P) \leq 1 + D$ if $D = \max\{f(\mu(xy)) | xy \in E\}$. Also $d_{d,f,g}(P) = n - 1$.

**Proof:** If $D = f(\mu(v_i v_{i+1}))$ then the vertex $v_i$ is colored by color 1 and the vertex $v_{i+1}$ is colored by $1 + D$. Also for coloring other vertices we assign vertices $v_j, v_{j+1}$ for all $j > i + 1$ by color 1 and $1 + D$ respectively, and for all $j < i + 1$ by color $1 + D$ and 1 respectively, therefore the chromatic number of $P$ is $1 + D$. By greedy coloring. Since for coloring fuzzy path we determined an edge with maximum value scale function $f$, denoted as $D$ and coloring all vertices by color 1 and $1 + D$. Also we can color fuzzy Path by some colors such that the relation in condition (4) on definition 4 is true. Then we must choose $n - 1$ vertices for forcing color of the last vertex. So $d_{d,f,g}(P) = n - 1$.

**Definition 11** A fuzzy graph $G = (\sigma, \mu)$ is said to be bipartite if the vertex set $V$ can be partitioned into two nonempty sets $V_1$ and $V_2$ such that $\mu(v_1 v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(xy) = \sigma(x) \land \sigma(y)$ for all $x \in V_1$ and $y \in V_2$ then $G$ is called a fuzzy complete bipartite graph and is denoted by $K_{\sigma_1, \sigma_2}$ where $\sigma_1$ and $\sigma_2$ are, respectively, the restrictions of $V$ to $V_1$ and $V_2$ [2].

**Proposition 12** Let $K_{\sigma_1, \sigma_2}$ be a fuzzy complete bipartite graph, $d$ a dissimilarity measure and $f, g$ be scale functions. where $g(\sigma_1(i), \sigma_2(j)) = f(\mu_{ij})$
for all \( i, j \in V \). Then \( 1 \leq \chi_{d,f,g}(K_{\sigma_1,\sigma_2}) \leq 1 + D \) if \( D = \max\{f(\mu(xy)) \mid xy \in E\} \). Also \( d_{d,f,g}(K_{\sigma_1,\sigma_2}) = n - 1 \).

**Proof:** Consider the greedy coloring. If \( D = f(\mu(uv)) \) where \( u \in V_1 \) and \( v \in V_2 \) then the vertex \( u \) is colored by color \( 1 + D \) and the vertex \( v \) is colored by \( 1 + D \). For coloring vertices we assign the color 1 to the vertices \( u_1 \) and the color \( 1 + D \) to the vertices \( v_j \). (Also we can color this by some colors such that the relation in (4) is true) therefore the chromatic number of \( K_{\sigma_1,\sigma_2} \) is \( 1 + D \). By the definition of \( K_{\sigma_1,\sigma_2} \), \( D = \max\{f(\mu(xy)) \mid xy \in E\} \) by the greedy coloring. Since we can color vertices of part one in \( K_{\sigma_1,\sigma_2} \) by color 1 and vertices part two by color \( 1 + D \). Also we can color this vertices by some colors such that the relation in condition (4) on definition 4 is true. Therefore we must choose \( n - 1 \) vertices for forcing the chromatic number \( K_{\sigma_1,\sigma_2} \). So \( d_{d,f,g}(K_{\sigma_1,\sigma_2}) = n - 1 \).

**Proposition 13** Let \( G_1 = (\sigma_1,\mu_1) \) and \( G_2 = (\sigma_2,\mu_2) \) be two fuzzy graphs on \( V_1 \) and \( V_2 \) respectively with \( V_1 \cap V_2 = \phi \), \( d \) a dissimilarity measure and the \( f, g \) be scale functions, where \( g(\sigma_1(i),\sigma_1(j)) = f(\mu_1(i)) \) for all \( i, j \in V \), \( t = 1, 2 \). If \( G = (G_1 \cup G_2) \) then \( 1 \leq \chi_{d,f,g}(G) \leq \max\{\chi_{d,f,g}(G_1),\chi_{d,f,g}(G_2)\} \). Also \( d_{d,f,g}(G_1 \cup G_2) = d_{d,f,g}(G_1) + d_{d,f,g}(G_2) \).

**Proof:** Let \( D = \max\{\chi_{d,f,g}(G_1),\chi_{d,f,g}(G_2)\} \). We consider the greedy coloring. If the fuzzy graph \( G_1 = (\sigma_1,\mu_1) \) is colored by \( \chi_{d,f,g}(G_1) \) colors and \( G_2 = (\sigma_2,\mu_2) \) is colored with \( \chi_{d,f,g}(G_2) \). By the definition of \( G = (G_1 \cup G_2) \), we color vertices \( v_i \in V_1 \) by \( D \) colors and coloring vertices \( v_j \in V_2 \) with \( D \) colors, therefore the chromatic number of \( G_1 \cup G_2 \) is \( D \). If the defining number of fuzzy graph \( G_1 = (\sigma_1,\mu_1) \) is \( d_{d,f,g}(G_1) \) and the defining number of \( G_2 = (\sigma_2,\mu_2) \) is \( d_{d,f,g}(G_2) \) then we can choose the vertices of \( G_1 \) and the vertices of \( G_2 \) therefore the defining number of \( G_1 \cup G_2 \) is obtained \( d_{d,f,g}(G_1) + d_{d,f,g}(G_2) \).

**Proposition 14** Let \( G_1 = (\sigma_1,\mu_1) \) and \( G_2 = (\sigma_2,\mu_2) \) be two fuzzy graphs on \( V_1 \) and \( V_2 \) respectively with \( V_1 \cap V_2 = \phi \), \( d \) a dissimilarity measure and the \( f, g \) be scale functions, where \( g(\sigma_1(i),\sigma_1(j)) = f(\mu_1(i)) \) for all \( i, j \in V \), \( t = 1, 2 \). If \( G = (G_1 \cup G_2) \) then \( 1 \leq \chi_{d,f,g}(G) \leq \max\{\chi_{d,f,g}(G_1),\chi_{d,f,g}(G_2)\} + D \). Where \( D = \max\{f(\mu(xy)) \mid xy \in E\} \).

**Proof:** We consider the greedy coloring. If the chromatic number of fuzzy graph \( G_1 = (\sigma_1,\mu_1) \) is \( \chi_{d,f,g}(G_1) \) and the chromatic number fuzzy graph \( G_2 = (\sigma_2,\mu_2) \) be two fuzzy graphs on \( V_1 \) and \( V_2 \) respectively with \( V_1 \cap V_2 = \phi \), \( d \) a dissimilarity measure and the \( f, g \) be scale functions, where \( g(\sigma_1(i),\sigma_1(j)) = f(\mu_1(i)) \) for all \( i, j \in V \), \( t = 1, 2 \). If \( G = (G_1 \cup G_2) \) then \( 1 \leq \chi_{d,f,g}(G) \leq \max\{\chi_{d,f,g}(G_1),\chi_{d,f,g}(G_2)\} + D \).
can assume that $D = f(\mu(u_1u_2))$ then the first vertex colored by color 1 and the second vertex is colored by $1 + D$ and the other vertices color by $1, 1 + D$ respectively. Since $n$ is even the $n_{th}$ vertex colored by $1 + D$ therefore its chromatic number in this case is $1 + D$. If assume that $n$ is odd, if $D = f(\mu(u_1u_{i+1}))$, without loss generality we can assume that $D = f(\mu(u_1u_2))$ then the first vertex is colored by color 1 and the second vertex is colored by $1 + D$ and the other vertices are colored by 1 and $1 + D$ respectively. Since $n$ is odd, the $(n-1)_{th}$ vertex colored by $1 + D$ and can’t color the $n_{th}$ vertex by color 1 because its neighborhood has color 1 therefore we color it by color $1 + 2D$. So the chromatic number in this case is $1 + 2D$. □

![Fig.1. Fuzzy cycles $\tilde{C}_6$ and $\tilde{C}_5$](image)

**Definition 18** A $n \square m$ fuzzy torus is defined as $\tilde{C}_n \square \tilde{C}_m$, where $\tilde{C}_n$ and $\tilde{C}_m$ are fuzzy cycles of length $n$ and $m$, respectively, and the multiplication is the fuzzy Cartesian product for graphs.

In what follows, the $m$ copies of $\tilde{C}_n$ in the fuzzy Cartesian product [10] are the rows of the torus, and the $n$ copies of $\tilde{C}_m$ are the columns of the torus.

**Lemma 19** Let $\tilde{C}_n \square \tilde{C}_m$ be a fuzzy torus, $d$ a dissimilarity measure and the $f, g$ be scale functions, where $g(\sigma_i(i), \sigma_j(j)) = f(\mu_{ij})$ for all $i, j \in V, t = 1, 2$. Then $1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m, \tilde{C}_m) \leq 1 + D$ if $n, m$ are even and $1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m, \tilde{C}_m) \leq 1 + 2D$ if $n$ or $m$ is odd where $D = \max\{|f(\mu_{xy})| \mid xy \in E\}$.

**Proof:** If $n, m$ both are even, by definition of fuzzy Cartesian product we have the $m$ copies of $\tilde{C}_n$ in the rows and the $n$ copies of $\tilde{C}_m$ in the columns if by proof of proposition 8 we can color the first copy of $\tilde{C}_n$ in row where the vertex $v_{11}$ is colored by color 1 and color the first column in fuzzy torus where contains $v_{11}$ by colors $1, 1 + D$ respectively. Therefore in other rows one vertex is colored and other vertex in any rows is colored too. So we have $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + D$. If one of the $m, n$ are odd then by definition of fuzzy Cartesian product and without loss generality assume $n$ is odd and $n$ is even then the first vertex of first rows as denoted $v_{11}$ colored by color 1 and other vertices this row by way of proof proposition 8 is colored by $1 + D, 1$ respectively. Since $m$ is odd we can coloring the first column by $1, 1 + D, 1 + 2D$ respectively. Also the first vertex of other rows is colored so we can colored them by way in proof of proposition 8 therefore in this case $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) \leq 1 + 2D$. If $n, m$ both are odd by consider greedy coloring and fuzzy Cartesian product we colored the first copies of $\tilde{C}_n$ in rows by color $1, 1 + D, 1 + 2D$ by way in proposition 8 then the first vertex of first copies of $\tilde{C}_n$ in column is colored by 1 and other vertices this column is coloring by $\{1 + D, 1 + 2D\}$ therefore for coloring any vertices of fuzzy torus in this state since two vertices in above and left it is colored in the worst state if two vertices have different colors we can colored it vertex by another color. So we can colored fuzzy torus by $\{1 + D, 1 + 2D\}$ or $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m, \tilde{C}_m) \leq 1 + 2D$, in the worst state if all vertices membership value and edge membership value are equal then $\chi_{d,f,g}(\tilde{C}_n \square \tilde{C}_m) = 1 + 2D$, .. □

![Fig.2. Fuzzy torus $\tilde{C}_4 \square \tilde{C}_4$ and $\tilde{C}_4 \square \tilde{C}_3$ and $\tilde{C}_3 \square \tilde{C}_3$.](image)

**Definition 20** A fuzzy hypercube $\tilde{Q}_n$ of dimension $n$ is defined as $\tilde{Q}_{n-1} \square \tilde{K}_2$, where $\tilde{K}_2$ is a fuzzy complete graph with 2 vertices, and $\tilde{Q}_{n-1}$ is fuzzy hypercube of dimension $n - 1$ with $2^{n-1}$ vertices and $\tilde{Q}_1 = \tilde{K}_2$.

**Proposition 21** Let $\tilde{Q}_n$ be a fuzzy hypercube of dimension $n$, $d$ a dissimilarity measure and the $f, g$ be scale functions, Then $1 \leq \chi_{d,f,g}(\tilde{Q}_n) \leq 1 + D$ if $D = f(\mu(e)), e \in E[\tilde{K}_2]$.

**Proof:** By greedy coloring, if $D = f(\mu(e)), e \in E[\tilde{K}_2]$, Since $\tilde{Q}_1 = \tilde{K}_2$, for $n = 1$ we can color.
\( \tilde{Q}_n \) by color 1, 1 + D. For any \( n \geq 2 \) by definition Cartesian product we have \( 2^{n-1} \) copies of \( \tilde{K}_2 \) which membership value of vertices of \( \tilde{Q}_n \) less than membership value of vertices of \( \tilde{K}_2 \) and edge membership value of \( \tilde{Q}_n \) less than \( D \). Therefore we can color any vertices of \( \tilde{K}_2 \) by 1, 1 + \( D \) therefore we color fuzzy hypercube of dimension \( n \) where its chromatic number at last 1 + \( D \). In the worst state if all vertices membership value and edge membership value are equal then \( \chi_{d,f,g}(\tilde{Q}_n) = 1 + D \), (see Fig.3).

Fig.3. Fuzzy hypercube \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \) and \( \tilde{Q}_3 \).

**Definition 22** Let \( \tilde{C}_n \) is fuzzy cycle of length \( n \) and \( \tilde{P}_m \) is fuzzy path of length \( m \), the \( \tilde{C}_n \square \tilde{P}_m \) is the fuzzy Cartesian product of them.

**Proposition 23** Let \( \tilde{C}_n \square \tilde{P}_m \) be a fuzzy graph, \( d \) a dissimilarity measure and the \( f,g \) be scale functions, \( D_1 = \max\{f(\mu(xy)) \mid xy \in E[\tilde{C}_n]\} \), \( D_2 = \max\{f(\mu(xy)) \mid xy \in E[\tilde{P}_m]\} \). Then 
\[
1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{P}_m) \leq 1 + D \text{ if } n \text{ is even and } 1 \leq \chi_{d,f,g}(\tilde{C}_n \square \tilde{P}_m) \leq 1 + 2D \text{ if } n \text{ is odd where } D = \max\{D_1, D_2\}.
\]

In what follows, the \( m \) copies of \( \tilde{C}_n \) in the Cartesian product are the rows of the \( \tilde{C}_n \square \tilde{P}_m \), and the \( n \) copies of \( \tilde{P}_m \) are the columns of the \( \tilde{C}_n \square \tilde{P}_m \).

**Proof:** If \( n \) is even by Lemma 1, we can color the fuzzy cycle \( \tilde{C}_n \) by color 1, 1 + D and by definition of fuzzy Cartesian product the \( n \) copies of fuzzy path \( \tilde{P}_m \) are in the column where the first vertex of all is colored with the color the first cycle in rows. Therefore for any \( \tilde{P}_m \) we can color other vertices by 1, 1 + D or 1 + D, 1. So in this case the chromatic number is at 1 + D. now assume \( n \) is odd. By proposition 8 we can color the fuzzy cycle \( \tilde{C}_n \) by colors 1, 1 + D, 1 + 2D. Then we obtained the \( n \) copies of fuzzy path \( \tilde{P}_m \) where one of its vertices is colored. We can color it by two colors of \{1, 1 + D, 1 + 2D\} respectively. Therefore in this case chromatic number at last 1 + 2D. also in the worst case if all vertices membership value and edge membership value are equal then \( \chi_{d,f,g}(\tilde{C}_n \square \tilde{P}_m) = 1 + 2D \). \( \square \)

**References**