Investigation on the spectrum of graph $\mathcal{G}_l$

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Abstract: Let $\mathcal{G}_l$ be the graph obtained from K_l by adhering the root of isomorphic trees $\mathcal{T}$ to every vertex of K_l. In this paper we study the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ for all positive integer l and give some result about the spectrum of the adjacency matrix $A(\mathcal{G}_l)$.

Key–Words: Adjacency matrix, complete graph, spectrum

1 Introduction

Let G be a simple undirected graph on n vertices, and let $A(G)$ be a $(0,1)$-adjacency matrix of G. Since $A(G)$ is a real symmetric matrix, all of its eigenvalues are real.Without loss of generality, that they are ordered in non-increasing order, i.e.,

$$\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G),$$

and call them the spectrum of G, The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of G.

About the spectrum and the spectral radius of graphs, a great deal of investigation is carried out [1,2,3]. Specially, to the special graphs, for example [4] studied the spectral radius of bicyclic graphs with n vertices and diameter d, [5] studied the spectral radius of trees with fixed diameter.

Let $\mathcal{T}$ be an unweighted rooted tree of k levels such that in each level the vertices have equal degree. K_l be a complete graph on l vertices. Let $\mathcal{G}_l$ be the graph obtained from K_l by adhering the root of isomorphic trees $\mathcal{T}$ to every vertex of K_l. Similar to the definition of tree's level, we agree that the complete graph K_l is at level 1, and that $\mathcal{G}_l$ has k levels. Thus the vertices in the level k have degree 1.

For $j = 1, 2, 3, ..., k$, Let n_{k-j+1} and d_{k-j+1} be the number of vertices and the degree of them in the level j. Observe that $n_k = l$ is the number of vertices in level 1 and n_1 the number of vertices in level k(the number of pendant vertices). Then,

$$n_{k-1} = (d_k - l + 1)n_k,$$

$$n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}, j = 2, 3, ..., k-1$$

Observe that d_k is the degree of vertex of the complete graph K_l in $\mathcal{G}_l$, d_1 is the degree of the vertices in the level k, $n_k = l$. The total number of vertices in the graph $\mathcal{G}_l$ is

$$n = \sum_{j=1}^{k-1} n_j + l$$

In general,using the labels $n, n-1..., 1$, in this order, our labeling for the vertices of $\mathcal{G}_l$ is:

(1)First, we label the vertices of K_l with clockwise direction.

(2)For one of vertices of level $j(j = 1, 2, ..., k-1)$, the bigger its labeling is , then the vertex of level $j+1$ adjacent to it should be labeled first.

(3)Label from level 1 to level k in turn.

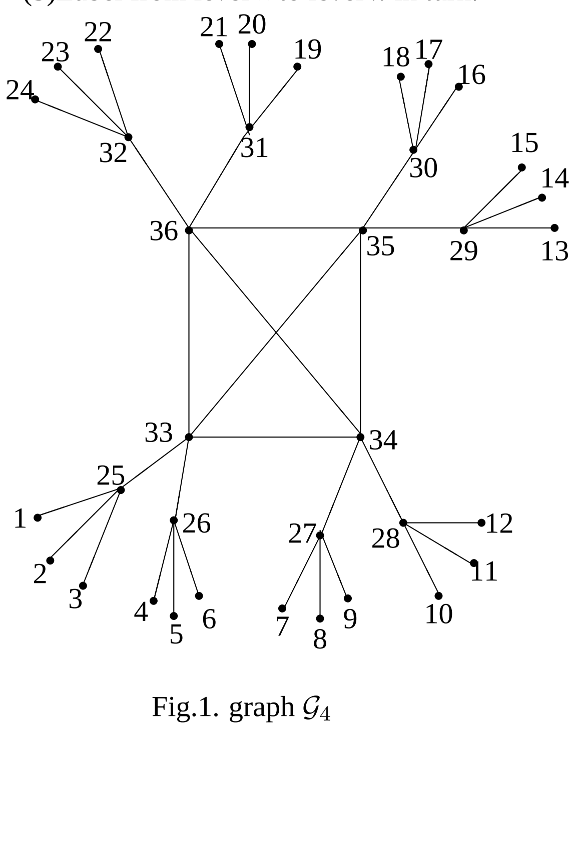

Fig.1. graph $\mathcal{G}_4$

Above(Fig.1.) is an example of a such graph $\mathcal{G}_4$ for $k = 3, n_1 = 24, n_2 = 8, n_3 = 4, d_1 = 1, d_2 = 4, d_3 = 5$.

[6], [7] studied the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ for case $l = 1$ and $l = 2$ respectively. In this paper we will study the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ for all positive integer l.

2 Preliminaries

We introduce the following notations:

(1) $\mathbf{0}$ is the all zeros matrix, the order of $\mathbf{0}$ will be clear from the context in which it is used.

(2)$\mathbf{I_m}$ is the identity matrix of order $m \times m$.

(3) $m_j = \frac{n_j}{n_{j+1}}$, for $j = 1, 2, ..., k-1$.

(4) $\mathbf{e_m}$ is the all ones column vetor of dimension m.

For $j = 1, 2, ..., k-1$, C_j is the block diagonal matrix

$$C_j = \begin{pmatrix} \mathbf{e_{m_j}} & & & \\ & \mathbf{e_{m_j}} & & \\ & & \ddots & \\ & & & \mathbf{e_{m_j}} \end{pmatrix}$$

with n_{j+1} diagonal blocks. Thus, the order of C_j is $n_j \times n_{j+1}$.

For example we use these notation with the graph $\mathcal{G}_4$ in Fig.1. $m_1 = \frac{n_1}{n_2} = 3, m_2 = \frac{n_2}{n_3} = 2$, then

$$C_1 = diag\{\mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}\},$$

$$C_2 = diag\{\mathbf{e_2}, \mathbf{e_2}, \mathbf{e_2}, \mathbf{e_2}\},$$

The adjacency matrix $A(\mathcal{G}_4)$ in Fig.1. become

$$A(\mathcal{G}_4) = \begin{pmatrix} \mathbf{0} & C_1 & \mathbf{0} \\ C_1^T & \mathbf{0} & C_2 \\ \mathbf{0} & C_2^T & B_4 \end{pmatrix}$$

where $B_4 = A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

In general, our labeling yields to

$$A(\mathcal{G}_l) = \begin{pmatrix} \mathbf{0} & C_1 & & & & \\ C_1^T & \mathbf{0} & C_2 & & & \\ & C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & C_{k-1} \\ & & & & C_{k-1}^T & B_l \end{pmatrix}$$

where $B_l = A(K_l) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$

Apply the Gaussian elimination procedure we obtained the following lemma:

Lemma 1. Let $M =$

$$\begin{pmatrix} \alpha_1 I_{n_1} & C_1 & & & & \\ C_1^T & \alpha_2 I_{n_2} & C_2 & & & \\ & C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \alpha_{k-1} I_{n_{k-1}} & C_{k-1} \\ & & & & C_{k-1}^T & \alpha_k I_{n_k} + B_l \end{pmatrix}$$

Let

$$\beta_1 = \alpha_1$$

and

$$\beta_j = \alpha_j - \frac{n_{j-1}}{n_j} \frac{1}{\beta_{j-1}}, j = 2, 3, ..., k, \beta_{j-1} \neq 0.$$

If $\beta_j \neq 0$ for all $j = 1, 2, ..., k-1$, then

$$detM = \beta_1^{n_1} \beta_2^{n_2} ... \beta_{k-1}^{n_{k-1}} (\beta_k - l + 1)(\beta_k + 1)^{l-1}. \quad (1)$$

Proof. Apply the Gaussian elimination procedure, without row interchanges, to M to obtain the block upper triangular matrix

$$\begin{pmatrix} \beta_1 I_{n_1} & C_1 & & & & \\ & \beta_2 I_{n_2} & C_2 & & & \\ & & \beta_3 I_{n_3} & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \beta_{k-1} I_{n_{k-1}} & C_{k-1} \\ & & & & & \beta_k I_{n_k} + B_l \end{pmatrix}$$

Hence,

$$detM = \beta_1^{n_1} \beta_2^{n_2} ... \beta_{k-1}^{n_{k-1}} det(\beta_k I_{n_k} + B_l),$$

since

$$det(\lambda I - B_l) = (\lambda - l + 1)(\lambda + 1)^{l-1},$$

so

$$detM = \beta_1^{n_1} \beta_2^{n_2} ... \beta_{k-1}^{n_{k-1}} (\beta_k - l + 1)(\beta_k + 1)^{l-1}.$$

Thus, (1) is proved. #

3 The spectrum of $A(\mathcal{G}_l)$

Let $D=$

$$\begin{pmatrix} -I_{n_1} &&&&& \\ & I_{n_2} &&&& \\ && -I_{n_3} &&& \\ &&& \ddots && \\ &&&& (-1)^{k-1}I_{n_{k-1}} & \\ &&&&& (-1)^k I_{n_k} \end{pmatrix}$$

we can easily see that

$$D(\lambda I + A(\mathcal{G}_l))D^{-1} = \lambda I - A(\mathcal{G}_l)$$

Let

$$\phi = \{1, 2, ..., k-1\}$$

and

$$\Omega = \{j \in \phi : n_j > n_{j+1}\}$$

Observe that $n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}, j = 2, 3, ..., k-1$ and $n_{k-1} = (d_k - l + 1)n_k$. Observe also that if $j \in \phi - \Omega$ then $n_j = n_{j+1}$ and C_j is the identity matrix of order n_j.

Theorem 1. Let

$$S_0(\lambda) = 1, S_1(\lambda) = \lambda,$$

$S_j(\lambda) = \lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda)$, for $j = 2, 3, ..., k-1$,

$$S_k^-(\lambda) = (\lambda+1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l}S_{k-2}(\lambda)$$

and

$$S_k^+(\lambda) = (\lambda - l + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l}S_{k-2}(\lambda).$$

Then

(i) If $S_j(\lambda) \neq 0$, for $j = 1, 2, ..., k-1$, then

$$det(\lambda I - A(\mathcal{G}_l)) = (S_k^-(\lambda))^{l-1} S_k^+(\lambda) \prod_{j\in\Omega} S_j^{n_j - n_{j+1}}(\lambda). \quad (2)$$

(ii)The spectrum of $A(\mathcal{G}_l)$ is $\sigma(A(\mathcal{G}_l)) = (\cup_{j\in\Omega}\{\lambda : S_j(\lambda) = 0\}) \cup \{\lambda : S_k^-(\lambda) = 0\} \cup \{\lambda : S_k^+(\lambda) = 0\}$.

Proof. Suppose $S_j(\lambda) \neq 0$ for all $j = 1, 2, ..., k-1$. We apply lemma 1 to $M = \lambda I + A(\mathcal{G}_l)$

$\lambda I + A(\mathcal{G}_l) =$

$$\begin{pmatrix} \lambda I_{n_1} & C_1 &&&& \\ C_1^T & \lambda I_{n_2} & C_2 &&& \\ & C_2^T & \ddots & \ddots && \\ && \ddots & \ddots & \ddots & \\ &&& \ddots & \lambda I_{n_{k-1}} & C_{k-1} \\ &&&& C_{k-1}^T & \lambda I_{n_k} + B_l \end{pmatrix}$$

We have

$$\beta_1 = \lambda = S_1(\lambda) \neq 0,$$

$$\begin{aligned} \beta_2 &= \lambda - \frac{n_1}{n_2}\frac{1}{\beta_1} = \lambda - \frac{n_1}{n_2}\frac{1}{S_1(\lambda)} \\ &= \frac{\lambda S_1(\lambda) - \frac{n_1}{n_2}S_0(\lambda)}{S_1(\lambda)} = \frac{S_2(\lambda)}{S_1(\lambda)} \neq 0 \end{aligned}$$

Similarly, for $j = 3, 4, ..., k-1, k$

$$\begin{aligned} \beta_j &= \lambda - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}} = \lambda - \frac{n_{j-1}}{n_j}\frac{S_{j-2}(\lambda)}{S_{j-1}(\lambda)} \\ &= \frac{\lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j}S_{j-2}(\lambda)}{S_{j-1}(\lambda)} = \frac{S_j(\lambda)}{S_{j-1}(\lambda)} \neq 0 \end{aligned}$$

Thus

$$\begin{aligned} \beta_k + 1 &= \frac{S_k(\lambda)}{S_{k-1}(\lambda)} + 1 \\ &= \frac{(\lambda+1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l}S_{k-2}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{S_k^-(\lambda)}{S_{k-1}(\lambda)}, \end{aligned}$$

$$\begin{aligned} \beta_k - l + 1 &= \frac{S_k(\lambda)}{S_{k-1}(\lambda)} - l + 1 \\ &= \frac{(\lambda - l + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l}S_{k-2}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{S_k^+(\lambda)}{S_{k-1}(\lambda)} \end{aligned}$$

Therefore, from Lemma 1,

$$\begin{aligned} & det(\lambda I + A(\mathcal{G}_l)) \\ =\ & S_1^{n_1}(\lambda)\frac{S_2^{n_2}(\lambda)}{S_1^{n_2}(\lambda)} \cdots \frac{S_{k-1}^{n_{k-1}}(\lambda)}{S_{k-2}^{n_{k-1}}(\lambda)} \frac{S_k^+(\lambda)}{S_{k-1}(\lambda)} \frac{(S_k^-(\lambda))^{l-1}}{S_{k-1}^{l-1}(\lambda)} \\ =\ & S_1^{n_1-n_2}(\lambda)S_2^{n_2-n_3}(\lambda)...S_{k-1}^{n_{k-1}-n_k}(\lambda) \\ & \times S_k^+(\lambda)(S_k^-(\lambda))^{l-1} \\ =\ & (S_k^-(\lambda))^{l-1}S_k^+(\lambda)\prod_{j\in\Omega} S_j^{n_j-n_{j+1}}(\lambda) \end{aligned}$$

Since $det(\lambda I - A(\mathcal{G}_l)) = det(\lambda I + A(\mathcal{G}_l))$. Thus (i) is proved. Similar to the proof in [7], we can get (ii) by (i) . #

Let R_k^+ and R_k^- be the $k \times k$ symmetric tridiagonal matrices

$R_k^+ =$

$$\begin{pmatrix} 0 & \sqrt{d_2-1} &&& \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} && \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ && \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k-l+1} \\ &&& \sqrt{d_k-l+1} & l-1 \end{pmatrix}$$

and $R_k^- =$

$$\begin{pmatrix} 0 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k-l+1} & \\ & & \sqrt{d_k-l+1} & -1 & \end{pmatrix}$$

Observe that

$$R_k^+ = R_k^- + diag\{0, 0, ..., 0, l\}.$$

Theorem 2. For $j = 1, 2, 3, ..., k-1$, let R_j be the $j \times j$ leading principal submatrix R_k^+. Then

$$det(\lambda I - R_j) = S_j(\lambda), j = 1, 2, ..., k-1,$$

$$det(\lambda I - R_k^-) = S_k^-(\lambda),$$

$$det(\lambda I - R_k^+) = S_k^+(\lambda).$$

Proof. It is well know [8] that the characteristic polynomials Q_j of the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$H = \begin{pmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{k-1} & b_{k-1} \\ & & & & b_{k-1} & a_k \end{pmatrix}$$

satisfy the tree-term recursion formula

$$Q_j(\lambda) = (\lambda - a_j)Q_{j-1}(\lambda) - b_{j-1}^2 Q_{j-2}(\lambda)$$

with

$$Q_0(\lambda) = 1 \quad \text{and} \quad Q_1(\lambda) = \lambda - a_1.$$

In our case, $a_1 = a_2 = ... = a_{k-1} = 0, a_k = l - 1$(or $\quad a_k = -1$) and

$$b_{k-1} = \sqrt{\frac{n_{k-1}}{n_k}} = \sqrt{d_k - l + 1},$$

$$b_j = \sqrt{\frac{n_j}{n_{j+1}}} = \sqrt{d_{j+1} - 1}$$
$$\text{for} \quad j = 1, 2, 3, ..., k-2.$$

For these values, the above recursion formula gives the polynomials $S_j(\lambda), j = 0, 1, 2, ..., k-1, S_k^+(\lambda)$ and $S_k^-(\lambda)$.

This completes the proof. #

Theorem 3. Let $R_j, j = 1, 2, ..., k-1, R_k^+$ and R_k^- as above. then

(i)$\sigma(A(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-)$.

(ii) The multiplicity of each eigenvalue of the matrix R_j, as an eigenvalue of $A(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j \in \Omega$, 1 for the eigenvalues of R_k^+ and $l-1$ for the eigenvalues of R_k^-.

Proof. (i) is an immediate consequence of Theorem 1 and Theorem 2. From the strict interlacing property[9] for a symmetric tridiagonal matrix with nonzero codiagonal entries, it follows that its eigenvalues are simple. Hence the eigenvalues of $R_j, j = 1, 2, ..., k-1, R_k^+$ and R_k^- are simply. Finally, we use (2) and theorem 2 to obtain(ii). #

Theorem 4. The largest eigenvalue of R_k^+ is the largest eigenvalue of $A(\mathcal{G}_l)$.

Proof. It can be proofed by the strict interlacing property immediately. #

For example, for the graph $\mathcal{G}_4$ in Fig.1.

$$R_3^+ = \begin{pmatrix} 0 & \sqrt{3} & \\ \sqrt{3} & 0 & \sqrt{2} \\ & \sqrt{2} & 3 \end{pmatrix}$$

$$R_3^- = \begin{pmatrix} 0 & \sqrt{3} & \\ \sqrt{3} & 0 & \sqrt{2} \\ & \sqrt{2} & -1 \end{pmatrix}$$

and $\Omega = \{1, 2\}$. The eigenvalues of $A(\mathcal{G}_4)$ in Fig.1. are the eigenvalues of R_1, R_2, R_3^+ and R_3^-, they are

R_1 :	0		
R_2 :	−1.7320	1.7320	
R_3^- :	−2.5139	−0.5720	2.0860
R_3^+ :	−1.9459	1.2521	3.6938

The spectral radius of $\mathcal{G}_4$ in Fig.1. is $\lambda_1(A(\mathcal{G}_4)) = 3.6938$

4 Conclusion

We studied the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ for all positive integer l with an effective way. Let $R_j, j = 1, 2, ..., k-1, R_k^+$ and R_k^- as in section 3. We found that:

(1)$\sigma(A(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-)$.

(2) The multiplicity of each eigenvalue of the matrix R_j, as an eigenvalue of $A(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j \in \Omega$, 1 for the eigenvalues of R_k^+ and $l-1$ for the eigenvalues of R_k^-.

(3)The spectral radius of R_k^+ is the spectral radius of $A(\mathcal{G}_l)$.

It is very convenient with conclusions (1),(2),(3) to calculate the spectrum of the adjacency matrix $A(\mathcal{G}_l)$.

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