Sampling-Reconstruction Procedure of Markov Chains with Continuous Time and with an Arbitrary Number of States

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Abstract- At the first time the statistical description of the Sampling-Reconstruction Procedure of Markov Chains with continuous time and with an arbitrary number of states is given. The analytic expression for the conditional probability density of the jump time moment is obtained. The methodology of the sampling interval choice is suggested. One illustrative example is considered.

Key-Words. Sampling, reconstruction, error, Markov chain with continuous time

1 Introduction

The list of references dedicated to the problem of the statistical description of Sampling-Reconstruction Procedure (SRP) of stochastic processes is rather huge. Nevertheless, the SRP of Markov Chains with continuous time is not investigated in the literature. Here we notice paper [1] concerned with binary processes and two certificates of inventions [2, 3]. The mathematical models of Markov chains with continuous time are intensively used in the description of some real stochastic processes with jumps, for instance, impulse noise. This is the reason that it is necessary to know: how to sample, how to reconstruct and how to calculate the reconstruction errors of such processes. Naturally, the Markov chains with continuous time and with limited number of states are non continuous and non Gaussian processes. So, the usual method of the SRP investigation of continuous stochastic processes (i.e. the method of the conditional mathematical expectation rule) can not be applied directly. The features of the Markov chains with continuous time are very specify and the SRP investigation method must be another.

First, we need to take into account that the sampled realization of Markov chains with continuous time keeps its sampled value on the right and on the left of the sampling time during some random intervals. It means that we have to estimate the jump moment between two known different samples. In other words it is necessary to estimate *one* random variable during a sampling interval. (At the same time in the reconstruction of a continuous process we need to estimate random variables *permanently* at any instant moment between two neighbor samples.) Therefore the reconstruction error of a jump process will be characterized by the variance of the jump time estimation. This is the *first* specific feature of the SRP of the discussed processes. The *second* feature is connected with the absence of the problem of a reconstruction function because the shape of the reconstruction function is known: this is a straight line on the right and on the left of any known sample.

The present paper is the generalization of the recent publication [4] devoted to the SRP of a particular case of the Markov chains with continuous time and with *two* states. Here the number of states is an arbitrary. It is happened that this generalization is not trivial.

2 Problem Formulation

Let us have a Markov chain $\xi(t)$ with continuous time and with the states *0, 1, 2,…, N*. As is well known this chain is completely described by the rates of *N* Poisson's flows $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_N$ and by the matrix of the transfer probabilities $P_{ij} (P_{ii} = 0)$ at the jumps instants. Let us designate $t_0, t_1, ..., t_n, t_{n+1}$ as sampling moments. Let us $\xi(t_n) = i$. It is necessary to find the time interval T_i determined the next sampling instant $t_{n+1} = t_n + T_i$ if the variance of the estimation $\hat{\tau}_{ij}$ of the jump moment τ_{ij} from the state i into the state j $(j \neq i)$ is not more than a given value σ^2 (the same for all i and j):

$$V\hat{\tau}_{ij} \leq \sigma^2. \tag{1}$$

Because we deal with Markov processes the reconstruction characteristics inside of sampling intervals are depended on two neighbor samples only.

Below we shall find the estimation $\hat{\tau}_{ij}$, the variance $V\hat{\tau}_{ij}$ and the value of the interval T_i. In order to do this it is necessary to determine the conditional probability density function (pdf) of a jump moment between two arbitrary states.

3 The Choice of the Sampling Interval

In [4] we have proved that the conditional pdf of a jump moment between two known different samples in the binary Markov process is described by the cut exponential law. Using the same methodology and generalizing the problem for an arbitrary number of states one can find the following analytical result:

$$p(t \mid i, j) = C e^{-(\lambda_i - \lambda_j)t}, \quad 0 < t < T, \tag{2}$$

where

$$C = \begin{cases} \dfrac{\lambda_i - \lambda_j}{\left(1 - e^{-(\lambda_i - \lambda_j)T}\right)}, & if \quad \lambda_i \neq \lambda_j, \\ 1/T, & if \quad \lambda_i = \lambda_j. \end{cases} \tag{3}$$

The estimation $\hat{\tau}_{ij}$ for the jump moment τ_{ij} is the conditional mean:

$$\hat{\tau}_{ij} = E\{\tau_{ij} \mid i, j\} = \begin{cases} \dfrac{1}{\mu_{ij}}\left(1 - \dfrac{\mu_{ij}T}{e^{\mu_{ij}T} - 1}\right), & \mu_{ij} \neq 0, \\ T/2, & \mu_{ij} = 0, \end{cases} \tag{4}$$

where $\mu_{ij} = \lambda_i - \lambda_j$.

The variance of the estimation is:

$$V\hat{\tau}_{ij} = \begin{cases} \dfrac{1}{\mu_{ij}^2}\left[1 - \dfrac{(\mu_{ij}T)^2}{\left(1 - e^{-\mu_{ij}T}\right)\left(e^{\mu_{ij}T} - 1\right)}\right], & \mu_{ij} \neq 0,. \\ T^2/12, & \mu_{ij} = 0. \end{cases} \tag{5}$$

We notice that the conditional pdf (2) and the principal characteristics of the reconstruction procedure: the estimation (4) and the variance (5) do not depend on the transfer probabilities $P_{ij}\,(P_{ii} = 0)$ because the events (marked by indexes i and j) have been already realized.

Because $\xi(0) = i$ and the value $\xi(T_i)$ is unknown we have to choose the sampling interval T_i from the maximum of the variance:

$$\max_{j, i \neq j, P_{ij} \neq 0} V\hat{\tau}_{ij} = \sigma^2. \tag{6}$$

In order to find the maximum in (6) we need to take into account the states characterized by non zero transfer probabilities $P_{ij} \neq 0$. Other states can be ignored.

Because the variance (5) is the even function with regards of μ_{ij} and it decreases with the rise of $|\mu_{ij}|$ one can see that the maximum in (6) can be reached with the minimum of $|\mu_{ij}| = |\lambda_i - \lambda_j|$ (when j is changed).

Let us designate

$$\mu_i^* = \min_{j, i \neq j, P_{ij} \neq 0} \left|\lambda_i - \lambda_j\right|. \tag{7}$$

Then taking into account (5) the condition (6) will have the view:

$$\frac{1}{\mu_i^{*2}}\left[1 - \frac{\left(\mu_i^* T_i\right)^2}{\left(1 - e^{-\mu_i^* T_i}\right)\left(e^{\mu_i^* T_i} - 1\right)}\right] = \sigma^2. \tag{8}$$

Let us introduce two values x and z normalized with regards of time $1/\mu_i^*$:

$$x \equiv \frac{T_i}{1/\mu_i^*} = T_i \mu_i^*, \qquad z \equiv \frac{\sigma}{1/\mu_i^*} = \sigma \mu_i^*.$$

The equation (8) with respect of x is described in the form:

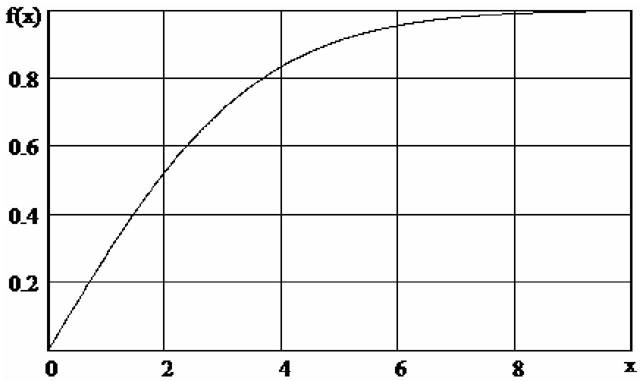

Fig. 1. The graph of the function $f(x)$.

$$\sqrt{1-\frac{x^2}{\left(1-e^{-x}\right)\left(e^{-x}-1\right)}}=z \qquad (9)$$

This expression determines the function $z = f(x)$ or

$\sigma\overset{*}{\mu}_i = f(T_i\overset{*}{\mu}_i)$.

The graph of this function is presented in Fig. 1. Because the values σ and $\overset{*}{\mu}_i$ must be known, therefore the sampling interval T_i is determined from the inverse function

$$T_i \equiv T_{\sigma i} = f^{-1}(\sigma\overset{*}{\mu}_i)/\overset{*}{\mu}_i. \qquad (10)$$

Putting (10) into (4) and (5) we obtain the required estimation of the jump moment (i.e. the reconstruction line) and the variance of the estimation (i.e. the reconstruction error).

4 Example

Let us consider the Markov chain with continuous time and with three states 0, 1 and 2. The rates of Poisson's flows are $\lambda_0 = 1, \lambda_1 = 0,1$ and $\lambda_2 = 0,05$. It means that this process is characterized by the following average times of staying in the states: 1, 10 and 20. Let us choose the matrix of the transfer probabilities in the view:

$$\boldsymbol{P} = \begin{bmatrix} 0 & P_{01} & 0 \\ P_{10} & 0 & P_{12} \\ 0 & P_{21} & 0 \end{bmatrix}.$$

The matrix elements indicate four possible transfers among the states. Let us choose $\sigma = 1$.

Following (7) we find the values $\overset{*}{\mu}_i$:

$$\overset{*}{\mu}_0 = 0.9, \quad \overset{*}{\mu}_1 = 0.05, \quad \overset{*}{\mu}_2 = 0.05.$$

Using the condition $\sigma = 1$ from (10) we determine the required sampling intervals:

$$T_0 = 5.36,\ T_1 = 3.47,\ T_2 = 3.47.$$

As one can see, generally the sampling intervals have different values. In this concrete example the samling interval for the state 0 has the length 5.36, and the intervals from the states 1 and 2 have the same length 3.47.

5 Conclusions

At the first time the statistical description of the SRP of Markov chains with continuous time and with an arbitrary number of states is carried out. The new methodology of the choice of the sampling intervals is suggested.

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