Second Order Systems for Filtering and Sampling Methods

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Abstract. This paper presents some invariance properties of differential equations, used for generating truncated test-function (functions which differ from zero only on a certain interval and with only some derivatives $f^{(1)}$, $f^{(2)}$, ..$f^{(n)}$ continuous on the real axis). The functions are used for the multiplication of some received signals, in order to take into account the random variations of the integration period of these signals, the integration being proportional to the mean value of the signal. Such fluctuations are generated by the switching phenomena at the end of the integration.
The paper presents also the properties of second order oscillating systems, considered as generating "practical" (i.e. truncated) test functions (PTF), in filtering and sampling procedures.

Key-Words: differential equations, truncated test-function, random variations, integration., second order oscillating systems, filtering and sampling

1.Introduction

The mean value of the received signal over a certain time interval is necessary in averaging procedures. This signal, being determined by devices with higher accuracy, is considered constant and the operation is performed by an integration of the signal on this time interval, using an electric current, which is charging a capacitor. For determining this time interval some random variations will appear due to the stochastic switching phenomena, for instance when the electric current charging the capacitor is interrupted. For this reason, a multiplication of the received signal with a test-function (a function which differs from zero only on this time interval and with continuous derivatives of any order on the whole real axis) is necessary. In the ideal case, such a test-function should have a form similar to a rectangular unity pulse defined on this time interval. However, such test functions (similar to the Dirac function) cannot be generated by a differential equation. The existence of such an equation of evolution, beginning to act at an initial moment of time, would imply the necessity for a derivative of certain order n (denoted $f^{(n)}$) to make a jump at this initial moment from the zero value to a nonzero value. But this is in contradiction with the property of the test-functions to have continuous derivatives of any order on the whole real axis, in this case represented by the time axis.

It results that an ideal test-function cannot be generated by a differential equation. For this reason, we must use "practical" (i.e. truncated) test functions (PTF) which differ from zero only on a certain interval and with only some derivatives $f^{(1)}$, $f^{(2)}$, ..$f^{(n)}$ continuous on the real axis). We shall show what properties should be satisfied by a differential equation of evolution in this case. We shall study also the filtering properties of such test functions, since the received signal usually presents a noise added to the useful component (considered to be continuous). It will be shown that very good results can be obtained by using an oscillating second order system.

2.Preliminaries

The study begins with the investigation of differential equations of evolution available for the multiplication with the received signal [1], making the average procedure be insensitive as related to the variations of the integration period. Devices with higher accuracy [2, 3] can decrease the effect of this variation (the switching noise), but they cannot lead to a substantial improvement as long as the mathematical model is not established.
The analysis begins by writing a test function under the form

$$\varphi = exp\ [1/(\tau^2 - 1)] \qquad (1)$$

where $\tau = t - t_{sym}$ (t_{sym} is the middle of the integration period). For $\tau \in [-1, 1]$, one can consider the function φ having nonzero values.

We are looking for a differential equation, which can have as solution a PTF.

By numerical simulation (using equations Runge-Kutta of 4-5 order in MATLAB) it has been obtained as solution a function f having a form similar to φ, but with a very small amplitude [1].

Continuing the analysis by studying a second order differential equation without free term, which has as possible solution the function φ, the numerical simulation (using the same Runge-Kutta functions in MATLAB) presents as solution a function with a form similar to φ, but still with a small amplitude (the amplitude is only four times greater than the amplitude obtained for a first order differential equation).

Then we try to obtain a function similar to a rectangular unit pulse. For this purpose, we consider a test function having the form

$$\varphi_a = exp[0.1/(\tau^2-1)] \qquad (2)$$

Using a second order differential equation (without free term) under the form

$$f^{(2)}(\tau) = [(0.6\tau^4 - 0.36\tau^2 - 0.2)/(\tau^2-1)]\ f(\tau)$$
(3)

(suggested by the expressions of φ_a, $\varphi_a^{(2)}$) and with initial conditions for f, $f^{(2)}$ equal to the values of φ_a, $\varphi_a^{(a)}$ at the initial time moment $\tau = -1 + 0.01$, we obtain as solution (using the same Runge-Kutta functions in MATLAB) a function very close to a rectangular unit pulse (the amplitude is close to the unit for more than 2/3 of the integration period).

By using the same procedure like in [1],
it results that for obtaining PTF we must use equations having the form:

$$d^{(2)}f\ /\ d\tau^2 = a(\tau^2)\ f \qquad (4)$$

or the form:

$$df\ /\ d\tau = \tau\, b(\tau^2) \qquad (5)$$

All the differential equations presented in this paragraph and studied by numerical simulation belong to one of these two classes of differential equations.

3.Filtering Properties of Practical Test Functions

One studies the behavior of such differential equations with a free term, which can be represented by an external signal u. In this case, the differential equation that should generate the PTF has the form:

$$g_n(\tau^2)\ df^{(n)}/d(\tau^2)^n + \ldots + g_0(\tau^2)\ f = u \quad (6)$$

on the time interval [-1, 1], with initial null conditions. It can be noticed that the solution f of this differential equation is also symmetrical as related to the point $\tau = 0$ if the signal u can be written also as a function of τ^2:

$$u = u(\tau^2) \qquad (7)$$

This condition is fulfilled if u is represented by a continuous received signal u, constant on the time interval [-1, 1]. It can be noticed that the equations

$$g_n(\tau^2)\ df_1^{\ (n)}/d(\tau^2)^n + \ldots + g_0(\tau^2)\ f_1 = u_1$$
$$g_n(\tau^2)\ df_2^{\ (n)}/d(\tau^2)^n + \ldots + g_0(\tau^2)\ f_2 = u_2$$
(8)

imply that $f = f_1 + f_2$ is a solution of the equation:

$$g_n(\tau^2)\ df^{(n)}/d(\tau^2)^n + \ldots + g_0(\tau^2)\ f = u \quad (9)$$

where $u = u_1 + u_2$. This means that we can study the behavior of the system when the input (the free-term) is represented by the continuous (useful) signal and when it is represented by the noise (an alternating signal), and then simply adds the results so as to obtain the output of the system when the noise is overlapped to the useful signal.

We begin by studying the system represented by the differential equation:

$$df\ /\ d(\tau^2) = f + u \qquad (10)$$

(the term f being used for stability). This can be written also:

$$df\ /\ [2\tau\ (d\tau)] = f + u \quad (11)$$

or

$$df\ /\ d\tau = 2\tau f + 2\tau u \qquad (12)$$

For u = 1 (the useful signal) and for initial null conditions, a function f which returns to zero at $\tau = 1$ is obtained; simulations performed by Runge-Kutta equations (in MATLAB) have shown an attenuation of about A = 3 for u having the form of $\sin 10\tau$. It has been also noticed that the mean value of the output oscillations generated in these circumstances differs from zero. Thus, the influence of the oscillations cannot be rejected by integration on the time interval [-1, 1]. The same aspects have been noticed for an input represented by the function $\sin(100\tau + \varphi)$; moreover, it has

been observed that the mean value of the output depends on φ (the initial phase of the input).
We try now to use a simpler differential equation, without the term “f”:

$$df / d(\tau^2) = u \quad (13)$$

which can be written as:

$$df / [2\tau(d\tau)] = u$$
$$df / d\tau = 2\tau u \quad (14)$$

The numerical simulations (performed by Runge-Kutta equations in MATLAB) have shown similar aspects to those noticed in the previous case. The form of the output (in the conditions of a unit-step input, u = 1) and the attenuation of an alternating component (in the conditions of u = sin100τ) are almost the same; the mean value of the output oscillations (generated by an input of frequency 100) still differs from zero and so this structure cannot be used for rejecting the influence of an alternating noise, added to the useful continuous signal.
Studying the second order systems continues. First, we write the derivative $d^2 f / d(\tau^2)^2$ under the form

$$d^2 f / d(\tau^2)^2 = d / d(\tau^2) [df / d(d\tau^2)] =$$
$$[1 / (2\tau)] d / d\tau \{[1 / (2\tau)] df / d\tau\} =$$
$$[1/(4\tau^2)] d^2f / d\tau^2 + [1 / (2\tau)] (df / d\tau) d / d\tau [1 / (2\tau)] =$$
$$[1 / (4\tau^2)] d^2f / d\tau^2 - [1 / (4\tau^3)] df / d\tau \quad (14)$$

In order to obtain a simpler differential equation at the previous relation for $d^2f / d(\tau^2)$, one can add a term proportional to $df / d(\tau^2)$ (and also equal to $[1 / (2\tau)]\, df / d\tau$) so that the second term in the expression of $d^2f / d(\tau^2)^2$ would disappear. So we must add a term written as $[1 / (2\tau^2)]\, df / d(\tau^2)$ and finally we can write:

$$d^2f / d(\tau^2)^2 + [1 / (2\tau^2)] df / d(\tau^2) = [1 / (4\tau^2)] d^2f / d\tau^2 \quad (15)$$

We multiply this relation by $4\tau^2$ (so as to obtain constant coefficients) and we also add a term “f” in order to avoid a parabolic output of the system in the conditions of a unit-step input. It results:

$$d^2f / d\tau^2 + f = u \quad (16)$$

The output of the system is symmetrical as related to the point τ = 0 for a step input, while the differential equation can be written as

$$4\tau^2 d^2f / d(\tau^2)^2 + 2df / d(\tau^2) + f = u \quad (17)$$

(we note that it depends on τ^2). But the equation (16) corresponds to an oscillating second order system. It results that we can continue our analysis by considering in the most general case a linear second order system, having the transfer function:

$$H(s) = 1 / [T_0 s^2 + 1] \quad (18)$$

Beginning to work from initial null conditions at the zero moment of time (the time origin can be translated from –1 to zero, since the coefficients of the differential equation are independent on time). In the conditions of a unit-step input, the output of the system has the form of $1 - \cos (\tau / T_0)$, being equal to zero (together with its derivative) at the moment of time $2\pi T_0$. As it can be noticed, the output has the form of a practical test function. Thus an integration of this output on the time interval $(0,\ 2\pi T_0)$ is practically insensitive at the switching phenomena appearing at the sampling moment of time.
Analyzing the influence of the oscillating system upon an alternating input of angular frequency ω, we can observe that the oscillating system attenuates about $(\omega/\omega_0)^2$ times such an input (ω_0 being equal to $1/T_0$). The integration on the time interval [0, 2πT0] leads to a supplementary attenuation of about $[(1/(2\pi)((\omega/\omega_0)]$ times. The oscillations with the form:

$$y_{osc} = a\sin(\omega_0 t) + b\cos(\omega_0 t) \quad (19)$$

generated by the input-alternating component, have lower amplitude and give a null result after integration over the time interval $[0,\ 2\pi T_0]$. So the simplest structure generating a PTF, in the conditions of a unit-step input and providing a very good attenuation for the influence of an alternating input component (the noise), is represented by an oscillating second order system, beginning to work at initial null conditions on a time period $[0,\ 2\pi T_0]$.

5 Experimental results

At the beginning of our experimental studies, an oscillating second order system for processing the received electrical scheme has been made, using an operational amplifier for lower frequency, with resistors R_0 connected at the (-) input and capacitors C_0 connected between the output and the (-) input (the well-known negative feedback); no resistors and capacitors were connected between the (+) connection and the “null” (as required by the necessity of compensating the influence of the polarizing currents at the input of the amplifiers). The output of the oscillating system has been integrated over a period using a similar device (based on an operational amplifier with a resistor R_i connected at the (-) input and a capacitor C_i connected on the negative feedback loop), at the end of the period the integrated signal being sampled. Even in such conditions, of a very simple electrical scheme, for a unit step input, the output of the oscillating system was less than 0.1 after a period. This means that the amount integrated at the sampling moment of time has been decreased 10 times (by an order of magnitude), using a very simple device based on an oscillating system, as compared to the case when an asymptotically stable second order system would have been used. Moreover, such an electrical scheme is a robust structure as related to the variation of temperature. The time constants T_i – for the integrating system- and T_0 –for the oscillating system –have the form

$$T_i = R_iC_i, \quad T_0 = R_0C_0 \qquad (20)$$

If the resistors R_0, R_i and the capacitors C_0, C_i are made from the same material, the coefficient for temperature variation will be the same. Thus the ratio

$$A(2\pi T_0)/T_i = A(2\pi R_0C_0)/(R_iC_i) = = 2\pi A(R_0/R_i)(C_0/C_i) \quad \text{`}(21)$$

(representing the result of the integration) is insensitive at temperature variations.

This electrical scheme will be improved in two main directions: by adding some elements for decreasing the output of the oscillating system at the time moment $2\pi T_0$ and by replacing the operational amplifiers with active elements working at higher frequencies, in order to increase the working frequency.

6.Conclusions

The new aspects of this tackling way consist in investigating step-by-step and in detail the invariance properties of the differential equations used for generating PTF (functions which differ from zero on a time interval and which possess some continuous derivatives $f^{(1)}, f^{(2),} ...f^{(n)}$ on the whole real axis). Such a function must multiply the received signal before this signal is integrated on this time interval, so that the result of the integration should be insensitive as related to the random variations of the integration period (variations caused by the switching phenomena). It has been shown that a "mirror" symmetry of a PTF as related to the middle of this time interval, implies the necessity for the function f to be written under the form

$g(\tau^2)$, τ being the difference between t and t_{sym} (the middle of the time interval). Numerical simulations investigated some models of second order differential equations for generating such functions and it was shown that the best results (a pulse similar to a unit rectangular pulse) were obtained using the differential equation:

$$f^{(2)}(\tau) = [(0.6\tau^4 - 0.36\tau^2 - 0.2)/(\tau^2-1)]\, f(\tau) \qquad (22)$$

with initial conditions chosen in a suitable manner. The filtering properties of such PTF have been also studied, and it was shown that the simplest structure generating PTF, in the conditions of a unit-step input and providing a very good attenuation for an alternating input component (the noise added to the useful signal) is represented by an oscillating second order system.

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