Axi-symmetric Deformation in Generalized Thermoelastic Diffusion

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Abstract: Solutions to the field equations in homogeneous, isotropic thermodiffusive elastic medium for two dimensional axisymmetric problem, in inverted transformed form has been obtained by using Laplace and Hankel transforms. As an application to normal force and thermal source are considered and results in thermoelastic medium as a particular case has also been discussed. Numerical results for displacements, stresses, temperature distribution and chemical potential distribution with normal stress $t_{zz}$, temperature distribution $\theta$, chemical potential distribution $\mu$ for copper material has been obtained and shown geometrically both in LS and GL theories of thermoelasticity.

Key words: Generalized thermoelastic diffusion, Axi-symmetric problem, Mechanical and thermal sources, Laplace and Hankel transforms.

1 Introduction

There are two different theories of generalized thermoelasticity: proposed by Lord and Shulman [1], Green and Lindsay [2]. LS theory is based on the modified Fourier laws of heat conduction and admits one relaxation time whereas the GL theory is based on modified energy equation and constitutive equations and allows two relaxation times.

The study of diffusion phenomenon is of great deal of interest due to its integrated circuit fabrication, “dopants” in controlled amounts into the semiconductor substrate. In particular, formation of the base and emitter in bipolar transistors, form integrated resistors, source/drain regions in Metal Oxide Semiconductor dope poly-silicon gates in MOS transistors. The phenomenon of diffusion is oftenly used in oil extractions.

Thermodiffusion in an elastic solid is due to coupling of the fields of temperature, mass diffusion and that of strain. Using the coupled thermoelastic model, Nowacki [3-6] developed the theory of thermoelastic diffusion and discussed dynamical problems of diffusion in solids. Olesiak and Pyryev [7] discussed a coupled quasi-stationary problem of thermodiffusion for an elastic cylinder. They studied the influences of cross effects arising from the coupling of the fields of temperature, mass diffusion and strain. Due to these cross effects, the thermal excitation results in an additional mass concentration generating the additional field of temperature. Genin and Xu [8] investigated a problem on thermoelastic plastic metals with mass diffusion.


So for various problems characterizing the axisymmetric disturbances have been investigated [14-17] in the present investigation.

2 Basic equations

Following, Lord and Shulman [1], Green and Lindsay[2] and Sherief et al.[9], the governing equations for isotropic homogeneous elastic solid with generalized thermoelastic diffusion in the absence of body forces and heat sources are:

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The constitutive relations
\[
\begin{align*}
\sigma_{ij} &= 2\mu e_{ij} + \delta_{ij}[\lambda e_{kk} - \beta_1(\theta + \tau_1 \dot{\theta})], \\
-\beta_2(C + \tau^1 \dot{C}) &+ \beta_2^1(C + \tau^1 \dot{C})] \\
P &= -\beta_2 e_{kk} \beta_1 e_{kk} + b(C + \tau^1 \dot{C}) - a(\theta + \tau_1 \dot{\theta})
\end{align*}
\] (1)

The equation of motion
\[
\mu u_{ij,ij} + (\lambda + \mu)u_{ij,ij} - \beta_1(\theta + \tau_1 \dot{\theta})u_{ij} - \beta_2(C + \tau^1 \dot{C})u_{ij} = \rho \ddot{u}_i 
\] (2)

The equation of heat conduction
\[
\rho C_e(\theta + \tau_0 \dot{\theta}) + \beta_1 T_0 (e + \Omega \tau_0 \dot{e})_i = aT_0 (C + \gamma \dot{C}) = K \Theta 
\] (4)

The equation of mass diffusion
\[
\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} = \frac{\rho c_0}{K} \frac{\partial C}{\partial t} 
\] (5)

where \( e_{ij} = \frac{1}{2}(u_{ij,j} + u_{ji,j}) \) (i, j = 1, 2, 3), \( \beta_1 = (3 \lambda + 2 \mu) \alpha_\tau \), \( \beta_2 = (3 \lambda + 2 \mu) \alpha_\chi \), \( \lambda, \mu \) Lamé’s constant; \( \alpha_\tau \) coeff. of linear diffusion; \( \alpha_\chi \) the coefficient of linear diffusion expansion; T is the absolute temperature; \( T_0 \) is the temperature in natural state such that \( T_0 / T < 1 \); \( \sigma_{ij} \) stress tensor, \( u_i \) displacement vector, \( \rho \) is the density assumed independent of time, \( e_{ij} \) strain tensor, P the chemical potential per unit mass, C concentration, \( C_e \) specific heat, K is the coefficient of thermal conductivity; D diffusion constant; \( \tau_0, \tau_1 \) thermal relaxation times; \( \tau^0, \tau^1 \) are the diffusion relaxation times a and b constant of diffusion and diffusive effects, respectively. The superposed dots denote derivative with respect to time. \( \delta_{ij} \) is the Kronecker’s delta.

3 Formulation and solution of the problem
Consider an isotropic, homogeneous elastic solid with generalized thermoelastic diffusion in the undeformed state at temperature \( T_0 \) in cylindrical polar co-ordinates system \((r, \phi, z)\) having origin on the surface \( z = 0 \) with the z-axis pointing normally into the medium. The problem being considered plane axisymmetric, we assume the displacement vector as:
\[
\ddot{u} = (u_r, 0, u_z),
\] (6)

(i.e. the field component \( u_\phi \) is zero and \( u_r, u_z \) and \( \theta \) are independent of \( \phi \))

To facilitate the solution, the following dimensionless quantities are introduced
\[
\begin{align*}
u_r &= \frac{\omega_r}{c_1}, \quad z' = \frac{\omega_r}{c_1} z, \quad t' = \frac{\omega_r}{c_1} t, \quad u_r = \frac{\omega_r}{c_1} u_r, \\
u_z &= \frac{\omega_z}{c_1}, \quad t_z = \frac{\omega_z}{c_1} t, \quad \gamma &= \frac{\omega_z}{c_1}, \\
\tau_1' &= \omega_1 \tau_1, \quad \tau_0' &= \omega_0 \tau_0, \\
P &= \frac{P}{\beta_2}, \quad P_1' = \frac{P_1}{\beta_1 T_0}, \quad P_2' = \frac{P_2}{\beta_1 T_0}
\end{align*}
\] (7)

where \( c_1^2 = \frac{\lambda + 2 \mu}{\rho}, \quad \omega_o = \frac{\rho C_e c_1^2}{K} \).

The expression relating displacement components \( u_r(r, z, t) \) and \( u_z(r, z, t) \) to the scalar potential functions \( \psi_1(r, z, t) \) and \( \psi_2(r, z, t) \) as
\[
\begin{align*}
\frac{\partial \psi_1}{\partial r} + \frac{\partial^2 \psi_2}{\partial r^2} \frac{\partial^2 \psi_2}{\partial z^2} + \frac{\partial^2 \psi_2}{\partial z^2} &= 0 \\
\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_z}{\partial r^2} \frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_z}{\partial z^2} &= 0
\end{align*}
\] (8)

Applying the Laplace and Hankel transform defined by
\[
\begin{align*}
\hat{f}(r, z, s) &= \int_0^\infty f(r, z, t) e^{-st} dt, \\
\hat{f}(\xi, z, s) &= \int_0^\infty f(r, z, s) r J_n(r \xi) dr
\end{align*}
\] (9)

On eqns. (3)- (5), after using eqns.(6)-(8) (suppressing the primes for convince) and eliminating \( \psi_1, \psi_2 \) from the resulting expressions, we obtain
\[
\begin{align*}
\frac{d^6}{dz^6} + Q \frac{d^4}{dz^4} + N \frac{d^2}{dz^2} + I \left( \psi_1, \psi_2, \dot{\psi}_2 \right) = 0
\end{align*}
\] (10)
\[
\left( \frac{d^2}{dz^2} - \lambda^2 \right) \tilde{\psi}_2 = 0,
\]
where \( Q = \frac{1}{E} [F - 3\xi^2 E] \), \( N = \frac{1}{E} [G - 2F\xi^2 + 3\xi^4 E] \), \( I = \frac{1}{E} [F\xi^4 - G\xi^2 + H - E\xi^6] \),
\[
\lambda^2 = \xi^2 + \frac{4a_4s^2}{a_3},
\]
and \( E = b_i a_{12} - a_6, \ H = -a_7 a_8 a_{13} \),
\[
F = -b_1 (a_1 a_{12} + a_1 + a_{10} a_{11}) + a_6 a_8 + a_7 (a_{12} + a_{10} a_{11}) + a_9 (a_{10} + a_9 a_{12}),
\]
\[
G = b_i a_4 a_{13} + a_7 (a_4 a_{12} + a_{13}) - a_5 a_6 a_{13},
\]
\[
a_3 = \frac{\mu}{\lambda + \mu}, \quad a_4 = \frac{\rho c_i^2}{\lambda + \mu}, \quad b_1 = (1 + a_3),
\]
\[
a_5 = a_4 (1 + \tau_1 s), \quad a_6 = a_4 (1 + \tau_1 s), \quad a_7 = a_4 s^2,
\]
\[
a_8 = (1 + \tau_1 s)s, \quad a_9 = (1 + \Omega \tau_0 s)s,
\]
\[
a_{10} = a_4 (1 + \tau_1 s), \quad a_{11} = (1 + \tau_1 s),
\]
\[
a_{12} = a_4 (1 + \tau_1 s), \quad a_{13} = a_4 (1 + \Omega \tau^0 s)s,
\]
\[
\varepsilon_1 = \frac{\beta_1^2 T_0}{\rho C_E \lambda + 2\mu}, \quad a_1 = \frac{a(\lambda + 2\mu)}{\beta_1 \beta_2},
\]
\[
\varepsilon_2 = \frac{b(\lambda + 2\mu)}{\beta_2^2}, \quad a_2 = \frac{1}{bD\eta}, \quad \eta = \frac{\rho C_E}{K}.
\]

The roots of the eqn. (10) are \( \pm \lambda_i \) \((i = 1,2,3)\) and the roots of eqn. (11) are \( \pm \lambda_4 \). Making use of radiation condition \( \tilde{\psi}_1, \tilde{\theta}, \tilde{C} \) and \( \tilde{\psi}_2 \to 0 \) as \( z \to \infty \), the solutions of equations (10) and (11) may be written as
\[
\tilde{\psi}_1 = A_1 e^{-\lambda_1 z} + A_2 e^{-\lambda_2 z} + A_3 e^{-\lambda_3 z},
\]
\[
\tilde{\theta} = d_1 A_1 e^{-\lambda_1 z} + d_2 A_2 e^{-\lambda_2 z} + d_3 A_3 e^{-\lambda_3 z},
\]
\[
\tilde{C} = e_1 A_1 e^{-\lambda_1 z} + e_2 A_2 e^{-\lambda_2 z} + e_3 A_3 e^{-\lambda_3 z},
\]
\[
\tilde{\psi}_2 = A_4 e^{-\lambda_4 z},
\]
where \( d_1 = \frac{P^* \lambda_1^2 + Q^*}{a_{10}^2 + R^*} \),
\[
e_1 = \frac{U^* \lambda_1^2 + V^* \lambda_1^2 + W^*}{X^* \lambda_1^2 + T^*}, (i = 1,2,3),
\]
\[
P^* = \left( \frac{b_i}{a_6} + a_9 \right) a_{10}, \quad Q^* = \left( \frac{b_i \xi^2 + a_7}{a_6} + \xi \frac{\xi}{a_{10}} \right),
\]
\[
R^* = -\left( \frac{(\xi^2 + b_i)}{a_6} + \frac{a_5}{a_{10}} \right) U^* = 1 + \frac{a_{11} b_i}{(-a_5)},
\]
\[
V^* = \left( -2\xi^2 \left( 1 - \frac{b_i a_{11}}{a_5} \right) + \frac{a_{11} a_7}{a_5} \right),
\]
\[
X^* = \frac{a_1 a_6}{a_5} - a_{12}, \quad W^* = -\frac{a_{11}}{a_5} (b_i \xi^4 + a_7 \xi^2),
\]
\[
T^* = a_{12} \xi^2 - \frac{a_1 a_6}{a_5} + a_{13}
\]
with \( A_i \) \((i = 1,2,3)\) being arbitrary constants.

4 Applications

Mechanical forces acting on the surface

The boundary conditions in this case are
\( (i) |_{z=0}(r,z,t) = -P_1(r,t), \quad (ii) \theta = 0, \quad (iii) \theta = 0, \quad (iv) P = 0 \) at \( z = 0 \)
where \( P_1(r,t) \) is a well behaved function.

Concentrated normal force

When plane boundary is subjected to concentrated normal force,
\[
P_i(r,t) = \frac{P_i \delta(r) \delta(t)}{2\pi r},
\]
P is the magnitude of force applied and \( \delta(t) \) is the Dirac delta function. Using eqns. (1), (2), (6)-(8) in the boundary condition (17) and applying the transforms and in the resulting equations, substitute the values of \( \tilde{\psi}_1, \tilde{\theta}, \tilde{C} \) and \( \tilde{\psi}_2 \) from eqns. (13)-(16), we obtain the expressions for the components of displacement, stress, temperature distribution and chemical potential distribution as
\[
\tilde{u}_r = \frac{1}{\Delta} \left[ \frac{P_{11}(\xi)}{2\pi} \left( \lambda_1 \Delta e^{-\lambda_1 z} - \lambda_2 \Delta e^{-\lambda_2 z} \right) \right],
\]
\[
\tilde{u}_z = \frac{1}{\lambda} \left[ \frac{P_{11}(\xi)}{2\pi} \left( \lambda_1 \Delta e^{-\lambda_1 z} - \lambda_2 \Delta e^{-\lambda_2 z} \right) \right],
\]
\[
\tilde{t}_z = \frac{1}{\Delta} \left[ \frac{P_{11}(\xi)}{2\pi} \left( \lambda_1 \Delta e^{-\lambda_1 z} - \lambda_2 \Delta e^{-\lambda_2 z} \right) \right],
\]
\[
\tilde{t}_{zz} = \frac{1}{\Delta} \left[ \frac{P_{11}(\xi)}{2\pi} \left( \lambda_1 \Delta e^{-\lambda_1 z} - \lambda_2 \Delta e^{-\lambda_2 z} \right) \right],
\]
\[\tilde{\theta} = \frac{1}{\Delta} \left\{ \frac{P_1}{2\pi} \left( d_1 \Delta_1 e^{-\lambda_1 z} - d_2 \Delta_2 e^{-\lambda_2 z} \right) \right\}, \quad (23)\]

\[\tilde{P} = \frac{1}{\Delta} \left\{ \frac{P_1}{2\pi} \left( t_1 \Delta_1 e^{-\lambda_1 z} - t_2 \Delta_2 e^{-\lambda_2 z} \right) \right\}, \quad (24)\]

where

\[\Delta = -(s_4 \lambda_1 + m_2 s_1)(d_2 t_3 - d_3 t_2) + (s_4 \lambda_2 + m_2 s_2)(d_1 t_3 - d_3 t_1) + (s_4 \lambda_3 + m_2 s_3)(d_1 t_2 - d_2 t_1),\]

\[\Delta_1 = m_2 (d_1 t_3 - d_3 t_2), \quad \Delta_2 = m_2 (d_1 t_3 - d_3 t_1),\]

\[\Delta_3 = m_2 (d_1 t_2 - d_2 t_1),\]

\[\Delta_4 = \lambda_1 (d_2 t_3 - d_3 t_2) - \lambda_2 (d_1 t_3 - d_3 t_1) + \lambda_3 (d_1 t_2 - d_2 t_1),\]

\[s_4 = -(b_2 \xi^2 + b_3) \lambda_4,\]

\[t_i = \xi^2 \lambda_i + \varepsilon_i (1 + \tau^i s)e_i - a_i (1 + \tau_i s)d_i, \quad (l = 1, 2, 3)\]

\[b_2 = \frac{2\mu + \lambda}{\beta_1 T_0}, \quad b_3 = -\frac{\lambda \xi^2}{\beta_1 T_0}.\]

5 Particular case

Neglecting diffusion effect i.e. \(\beta_2 = a = b = 0\) in eqns.(19)-(24), we obtain the corresponding expressions for components of displacement, stress and temperature distribution as:

\[\tilde{u}_r = \frac{1}{\Delta^*} \left\{ \frac{P_1}{2\pi} (-\xi) \left( -\Delta_1' e^{-\lambda_1 z} + \Delta_2' e^{-\lambda_2 z} \right) \right\}, \quad (25)\]

\[\tilde{u}_z = -\frac{1}{\Delta^*} \left\{ \frac{P_1}{2\pi} \left( -\lambda_1 \Delta_1' e^{-\lambda_1 z} + \lambda_2 \Delta_2' e^{-\lambda_2 z} + \xi^2 \Delta_4' e^{-\lambda_4 z} \right) \right\}, \quad (26)\]

\[\tilde{\tau}_{rz} = \frac{1}{\Delta^*} \left\{ \frac{P_1}{2\pi} \left( -\lambda_1 \Delta_1' e^{-\lambda_1 z} \right) \right\}, \quad (27)\]

\[\tilde{\tau}_{zz} = \frac{1}{\Delta^*} \left\{ \frac{P_1}{2\pi} \left( s_1 \Delta_1' e^{-\lambda_1 z} + s_2 \Delta_2' e^{-\lambda_2 z} - s_4 \Delta_4' e^{-\lambda_4 z} \right) \right\}, \quad (28)\]

\[\tilde{\theta} = \frac{1}{\Delta^*} \left\{ \frac{P_1}{2\pi} \left( -d_1 \Delta_1' e^{-\lambda_1 z} + d_2 \Delta_2' e^{-\lambda_2 z} \right) \right\}, \quad (29)\]

where

\[\Delta^* = m_2 s_1 d_2 - m_2 s_2 d_1 + s_4 (\lambda_1 d_2 - d_1 \lambda_2),\]

\[\Delta_1' = m_2 d_1, \quad \Delta_2' = m_2 d_1, \quad \Delta_3' = (\lambda_1 d_2 - d_1 \lambda_2),\]

\[s_l = b_2 (\lambda_l^2 - (1 + \tau_l s)d_l) + b_3, \quad (l = 1, 2)\]

\[s_4 = -(b_2 \xi^2 + b_3) \lambda_4,\]

\[b_2 = \frac{2\mu + \lambda}{\beta_1 T_0}, \quad b_3 = -\frac{\lambda \xi^2}{\beta_1 T_0}.\]

6 Special cases

(6.1) By putting \(\alpha = 0, \beta = 0, \tau_1 = 0, \tau_1' = 0, \Omega = 1, \gamma = \tau_0\) in eqns. (19)-(24) and (25)-(29), we obtain the corresponding expressions of thermoelastic diffusion and thermoelasticity, respectively, for LS theory.

(6.2) For GL theory, we obtain the corresponding expressions of thermoelastic diffusion and thermoelasticity, respectively, by substituting \(\alpha = \tau_0, \beta = \tau_0', \gamma = \tau_0' > 0, \Omega = 0, \tau_0, \tau_0' > 0\) in eqns. (19)-(24) and (25)-(29).

(6.3) In case of coupled thermoelasticity, the thermal relaxation times vanishes i.e. \(\tau_0 = 0, \tau_0' = 0, \tau_1 = 0, \tau_1' = 0\) and consequently, we obtain the corresponding expressions of thermoelastic diffusion and thermoelasticity.

7 Inversion of the transforms

To obtain the solution of the problem in the physical domain, we must invert the transforms in eqns.(19)-(24), for the LS and GL theories. These expressions are functions of \(z\), the parameters of Laplace transforms \(s\) and \(\xi\), respectively, and hence are of the form \(\tilde{f}(\xi, z, s)\). To get the function \(f(r, z, t)\) in the physical domain, first we invert the Hankel transform using

\[\hat{f}(r, z, s) = \int_0^{\infty} \xi \tilde{f}(\xi, z, s) J_\alpha(\xi r) d\xi, \quad (30)\]

Now, for the fixed values of \(\xi, z\) and \(r\), the \(\hat{f}(r, z, s)\) in the expression (above) can be considered as the Laplace transform \(\hat{g}(s)\) of \(g(t)\). Following Honig and Hirdes[18], the Laplace transformed function \(\hat{g}(s)\) can be inverted. The last step is to calculate the integral in equation (30). The method for evaluating this integral is described Press et al. [19], which involves the use of Romberg’s integration with adaptive step size. This also
uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

8 Numerical results and discussion

Following Sherief and Saleh [10] copper material is chosen for the purpose of numerical calculation. \( T_0 = 293 K \), \( \rho = 8954 \text{ Kg m}^{-3} \), \( C = 383.1 \text{ JKg}^{-1} \text{K}^{-1} \), \( \alpha_t = 1.78(10)^{-5} \text{ K}^{-1} \), \( \alpha_c = 1.98(10)^{-4} \text{ m}^2 \text{Kg}^{-1} \text{K}^{-1} \), \( K = 386 \text{ Wm}^{-1} \text{K}^{-1} \), \( \lambda = 7.76(10)^{10} \text{ Kgm}^{-1} \text{s}^{-2} \), \( \mu = 3.86(10)^{10} \text{ Kgm}^{-1} \text{s}^{-2} \), \( D = 0.85(10)^{-8} \text{ Kg} \text{s} \text{m}^{-3} \), \( a = 1.2(10)^{4} \text{ m}^2 \text{K}^{-1} \text{s}^{-2} \), \( b = 0.9(10)^{8} \text{ m}^5 \text{Kg}^{-1} \text{K}^{-1} \text{s}^{-2} \). With non-dimensional relaxation times taken as \( \tau_0 = 0.02 \), \( \tau^* = 0.2 \) and \( \tau_1 = 0.03 \), \( \tau^1 = 0.3 \).

The values of normal stress \( t_z \), temperature distribution \( \theta \) and chemical potential distribution \( P \) for LS theory with and without diffusion effect (LS and LSWD) and GL theory with and without diffusion effect (GL and GLWD) have been studied for concentrated normal force at time \( t = 0.5 \). The variations of the components with distance \( r \) are shown (a) solid line for LS and solid line with center symbol ‘triangle’ for GL (b) small dashed line for LSWD and small dashed line with center symbol ‘diamond’ for GLWD. The variations are shown in figures (1)-(3). The computations are carried out in the range \( 0 \leq r \leq 10 \).

Normal force on the boundary of half-space (Concentrated normal force)

Fig.1 shows the variations of normal stress \( t_z \) with distance \( r \). Due to the effect of diffusion, the values of \( t_z \) for both LS and GL theories with diffusion effect are more in comparison to without diffusion effect. However the trend of variation is oscillatory for both theories in both cases in whole range.

See fig. 2, near the point of application of source there is a sharp increase in the values of temperature distribution \( \theta \) for both LS and GL, reveals the effect of diffusion and have small variation near \( r = 0 \) whereas for LSWD and GLWD it shows small variations near \( r = 0 \) \( (0 \leq r \leq 10) \).

The values of chemical potential distribution \( P \) for LS theory are more than in comparison to GL theory in \( 0 \leq r \leq 1 \). As \( r \) increases further GL theory become more prominent in \( 1.3 \leq r \leq 2.7 \) and are close to each other in the remaining range with minor difference in magnitude (fig.3).

9 Conclusions

As \( r \) diverses from the point of application of source the component of normal stress and chemical potential distribution, follow an oscillatory pattern. As the disturbances travels through different constituents of the medium, it suffers sudden changes, resulting in a non-uniform pattern of curves. The trend of curves exhibits the properties of thermo-diffusivity of the medium and satisfies the requisite condition of the problem. The results of this problem are very useful in the two dimensional problem of dynamic response due to various sources of the thermoelastic diffusion which has the various geophysical and industrial application including oil extraction.

References


