Investigating Axial Flow between Eccentric Cylinders

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Abstract: - This paper continues the investigations of the axial flow between cylinders in slightly eccentric position where the axis of the inner cylinder is offset slightly with respect to the outer one. Such flow is also known as thread annular flow and its study is motivated by the fact that thread injection is a surgical technique that allows the injection of porous medical implants, consisting of synthetic biocompatible materials, into the body in a minimally invasive way. The porous thread is stored on a spool and injected within a fluid by applying a pressure gradient. As most of the fluid flow is transparent and the actual fluid motion is not readily apparent to the human eye, thus modeling of the problem is needed. Past research has contributed into modeling the thread injection process mathematically where the thread is in a slightly eccentric position. This paper presents further insights into the behavior of the flow by obtaining the friction factor, which is a dimensionless parameter that expresses the linear relationship between the pressure gradient along the cylinder and the mean flow velocity through the cylinder, of the obtained flow. With the value of the friction factor, the amount of the head loss or pressure drop along the cylinder can then be calculated. Discussions are mainly on the comparison made with experimental results as well as numerical results for flow between concentric cylinders.

Key-Words: axial flow, thread-annular flow, eccentric cylinders, boundary layer flow, surface friction

1 Introduction

The injection of fluid into the body by using a needle or syringe is an important application of fluid dynamics. Nowadays, a cosmetic plastic surgeon not only injects fluid into the body but also specially designed surgical threads that consist of synthetic biocompatible materials. Through modeling the problem mathematically, a better understanding of the fluid flow characteristics in a syringe can be obtained by looking at the actual flow patterns. Then, the injection process can be carried out more proficiently and the pain of patients can be reduced.

The thread injection process can be modeled mathematically by considering the axial flow between two cylinders where the inner cylinder represents the thread moving at a constant velocity and the outer cylinder represents the syringe. Past researches on this application considered the thread to be in a concentric position. Frei et al. [2] studied the problem experimentally and theoretically and they found that there is discrepancy between experimental and theoretical results. Walton [3] considered the stability of the basic thread-annular flow and found that the reported discrepancies could be due to the nonlinear instability of the obtained flow. In another study by Labadin et al [4], the thread considered is in a slightly eccentric position within the syringe. It was reported that there is a difference in the flow compared to that of the concentric case. The same authors in [5] discussed that the basic flow obtained must include higher order terms as shown in equation (1) so that the friction factor of the flow can be calculated. The friction factor is an important parameter as it is used to compare the works between [2] and [3]. Hence, the aim of this paper is to calculate the friction factor of the basic flow when the thread is in a slightly eccentric position and compare the results with that of [2] and [3].
2 Problem Formulation

The problem is formulated following the work in [4]. The basic flow of thread injection is obtained as below:

\[ u(r) = u_0(r) + \varepsilon \cos \theta u_1(r) + \varepsilon ^2 u_{20}(r) + \varepsilon ^2 \cos 2\theta u_{22}(r) + O(\varepsilon ^3) \]

\[ : u(r) = 1 - r^2 + \frac{V - 1 + \delta ^2}{\ln \delta} - \ln r + \varepsilon \cos \theta \left( \frac{1}{r} - r \right) \left( \frac{2\delta ^2}{(1 - \delta ^2) \ln \delta} - \frac{V - 1 + \delta ^2}{(1 - \delta ^2) \ln \delta} \right) \]

\[ + \varepsilon ^2 \left( \frac{\ln r}{(1 - \delta ^2) \ln \delta} \right) \left( 1 + \delta ^2 - \frac{V - 1 + \delta ^2}{\ln \delta} \right) \]

\[ + \varepsilon ^2 \cos 2\theta \left( r^2 - \frac{1}{r^2} \right) \left( \frac{V - 1 + \delta ^2}{2 \ln \delta} - \frac{2\delta ^2}{1 + \delta ^2} \right) + O(\varepsilon ^3) \]

Clearly, the basic flow depends on three parameters: the thread radius \( \delta \), which must be in the range \( 0 < \delta < 1 \) because the radius of the syringe is non-dimensionalized to unity; the thread injection velocity \( V \); and the eccentricity parameter \( \varepsilon \), which must be much smaller than \( \delta \) in our analysis.

We see that when \( \varepsilon = 0 \), the basic flow reduces to \( u = u_0(r) \) which is the concentric flow considered by Walton [3]. The friction factor \( \lambda \) is calculated using the formula

\[ \lambda = \frac{16\pi^2 (1 - \delta)(1 - \delta ^2) ^2}{RQ ^2} \]

based on the work of Walton [3]. Here, \( R \) is the pressure gradient-based Reynolds number and \( Q \) is the volumetric flow rate. Basically, friction factor is a dimensionless parameter that expresses the linear relationship between the pressure gradient along the cylinder and the mean flow velocity through the cylinder. It is also related to the Reynolds number of the flow and the degree of roughness of the cylinder’s inner surfaces. This definition implies that when the friction factor decreases, then the mean flow will increase. This may lead to turbulence depending on the viscosity of the fluid. Therefore in order to avoid the occurrence of turbulence, the friction factor must be low. In addition, by obtaining the friction factor, the amount of the head loss or pressure drop along the syringe can be calculated.

The obtained basic flow is not a function of \( r \) only but also in terms of \( \theta \), thus the volumetric flow rate \( Q \), which is the volume of fluid that passes through a given volume per unit time, is defined as

\[ Q = \int _0 ^1 u \sqrt {\gamma \delta} \rho d\theta \]  

So that on substitution of equation (1) into equation (2) and integrating the resulting equation, we have:

\[ : Q = \frac{\pi}{2} \int _0 ^1 \left[ 1 + \frac{(1 - \delta ^2)}{\ln \delta} - \delta ^2 - \frac{2\delta ^2}{\ln \delta} \right] V \]

\[ + \varepsilon ^2 \left( 1 + \delta ^2 - \frac{V - 1 + \delta ^2}{\ln \delta} \left( \delta ^2 - 2\delta ^2 \ln \delta - 1 \right) \right) \]

(3)

neglecting terms of \( O(\varepsilon ^3) \). We notice that the expression for the volumetric flow rate \( Q \) above is similar to that of [3]. The only difference is the second term in equation (3), which reflects the eccentricity term. The friction factor can now be calculated given values for thread radius \( \delta \), the thread injection velocity \( V \), the eccentricity \( \varepsilon \) as well as the pressure gradient-based Reynolds number \( R \). For consistency with the work of [2] and [3], \( R \) is redefined implicitly in terms of \( Re \) as in the equation (4) below:

\[ Re = \frac{2RQ}{\pi (1 + \delta)} \]  

(4)

where \( Re \) is the flux-based Reynolds number, which is used by [2].

3 Problem Solution

Here, our results will be based on the plots of \( \log_{10} |\lambda| \) versus \( \log_{10} Re \) in order to compare our results with that of [2] and [3].

In Figure 1, we plotted the friction factor of the obtained flow for \( \varepsilon = 0 \). To be consistent with the experimental work done in [2], \( \delta \) is set to 0.51 and \( V \) is kept to 0.178. The result obtained is superimposed with that of [3] depicted in Figure 1. As mentioned before, setting \( \varepsilon = 0 \) leads to flow in concentric case. Clearly, with \( \varepsilon = 0 \) our result agrees with that of [3] as shown in Figure 1.

In Figure 2, we plotted the coefficient of \( \varepsilon ^2 \) (in equation (3)) versus thread radius, \( \delta \) with various values of mean velocity, \( V \) and it shows that the coefficient of \( \varepsilon ^2 \)-term in equation (3) is always positive. This shows that the contribution from the \( \varepsilon ^2 \) term in the expression for \( Q \) will become significant when the eccentricity increases.
In order to compare quantitatively with the experimental results in [2], the plot $\log_{10} |\lambda| \text{ versus } \log_{10} Re$ is required. Figure 3 depicts this result with various values of eccentricity ($\varepsilon = 0$ to $\varepsilon = 0.4$), with $\delta$ and $V$ are once again kept to 0.51 and 0.178 respectively. The experimental results from [2] are also superimposed onto the figure. It is observed that the effect of the eccentricity is to decrease the friction factor and the effect becomes stronger when $\varepsilon$ is increased. This is due to the increasing of the flux from the positive contribution of the $\varepsilon^2$-term in equation (3) as seen in Figure (2).

Notice that the friction factor curve for the eccentric cases is always lower than that of concentric case ($\varepsilon = 0$). Besides that, in comparison of our results with the experimental results of [2], we can see that the experimental results are always lower than that of the concentric flow model and slightly away from the friction factor curve of the eccentric cases where the eccentricity value of each curve is different. According to MacDonald [1], the value of eccentricity is always equal to or less than $1 - \delta$. However, [2] never found that the mean eccentricity is equal to 1 in the experiments. In other words, there was never any touch between the thread and the wall of outer cylinder. Therefore, in this paper our eccentricity value is set to $\varepsilon < 1 - \delta$.

Figure 4 depicts the effects of increasing the thread radius $\delta$ and keeping the thread velocity $V$ set to 0.2 and the eccentricity is 0.15. Clearly, the friction factor shows a pronounced dependence on the thread radius. The effect of increasing the thread radius is to decrease the friction factor.
A similar effect is found when the thread velocity is increased and keeping the thread radius at the value 0.51 and eccentricity at 0.15. Here, our maximum of the velocity is set to $1 - \delta^2$ based on the work of [3]. As shown in Figure 5, the thread velocity has a significant decreasing effect on the friction factor. When the thread velocity is large enough, the friction factor will be decreased slightly.

4 Conclusion

In this paper, we have computed the friction factor for thread-annular flow where the thread is in a slightly eccentric position and also compared our results with the experimental results of [2] and the theoretical results of [3]. We have also investigated the behavior of the friction factor at different thread radius, injection velocities and eccentricities. Based on the investigations, we found that in all cases, the effect of the parameters is to decrease the friction factor by increasing the flux value. Besides that, it is shown that our results are analogous with that of [2] and it could be an indication that the discrepancy between the experiments and theoretical results is due to the effects of eccentricity of the thread. In terms of medical application, the results of our analysis suggest that it is desirable to use a small thread radius in order to maximize the friction factor during the injection process so that it will not lead to turbulence.

References:


