Computing exact symmetries of dynamical systems from their reduced system of equations can be interesting

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Abstract: The symmetry analysis of differential equations in the context of Lie point and nonlocal symmetries is rich in the literature. In this paper we present the computation of the exact symmetry transformations of dynamical systems from their reduced systems using the Kepler problem as vehicle. We also note that this computational technique is applicable to systems that can be reduced to couple oscillator(s) and a conservation law.

Key words: Exact, symmetries, Dynamical, systems, infinitesimal, generators, flow, Kepler, Lie.

1. Introduction

It is well known that symmetries in general and in particular Lie point symmetry analysis are formidable tools for finding solution to differential equations (ordinary or partial differential equations)[1,2,34,5]. Since Lie theory and more recent works/researches [2,4,6,7,8,9,10] in the subject, emphasis is placed on the achievement of the infinitesimal generators of the symmetries of the differential equations. This may be consequences associated with the actual computational complicities that are seen to be involved in obtaining the Lie symmetry transformations (Flow) given the fact that some symmetry transformations are nonlocal in their representations.[11,10,9,8] More so, the determinations of the symmetries of most physical dynamical systems actually posed significant challenges in the literature [10,11,12] as in the understanding of their physical properties, visa vise constants of the motion, first integrals, linearization and orbit equations. Noether theory is significant in the aspect of variational symmetries in that it provided a straightforward link between symmetries and constants of motions (first integrals).[1,2,3] It is also well known that in the case of the Kepler problem the Neother symmetries (five variational symmetries) are subset of the Lie symmetries obtained in the literature (by Krause, Nucci, etc). The events that followed the analysis of complete symmetry groups of differential equations brought to fore the reduction of order of dynamical systems.[11,13] This revolutionized the entire symmetry analysis although did not amount to deviation from the original idea of Lie but explicitly exposed the importance of nonlocal symmetries as bases for the actual integrability of differential equations.[7,11,12] The reduction of order algorithm reduced dynamical systems to systems of oscillator(s) and conservation laws, which admits Lie algorithm for determination of their symmetry generators. The applicability of the reduction of order algorithm is formidable for determining the Lie symmetry group of dynamical systems [6, 9, 10, 11,13]. We note here that the literature refers to the vector fields of the infinitesimal generators as symmetries. We have recently reported [14] that the exact symmetries of dynamical systems different from vector field of the infinitesimal generators of dynamical systems can be accurately computed from the Lie symmetry generators of their reduced systems obtained by the reduction of order algorithm. We also shown there that one could use analogous constants obtained from the Hamilton vector of dynamical systems instead of the Ermanno-Bernoulli constants to reduce dynamical systems to systems that admit Lie algorithm. In this paper we present the actual computations of these exact symmetry transformations of the Kepler problem. We shall only note that the same is true for generalized Kepler problem and the Kepler related problems such as MICZ for instance. In section 2, we present basic definitions that are crucial to the understanding of our discussion. Section 3 shall give the computation of the exact symmetry transformations of the Kepler problem in two-dimensions while section 4 treated three-dimensional case. In section 5 we shall present concluding remarks.

2. Basic definitions and concepts

Let $T : X \to X$ be a one-to-one, and onto mapping (transformation) defined on a sub-manifold $X \subset M$. The totality of such transformations $\tau(M)$ form a group where the composition of mappings plays the part of a group operation and the identity transformation is designated I_M. The point (x,t) to point $(\bar{x},\bar{t})$ transformation defined symmetry transformation in general conceptualization. If the point transformation depends on a group parameter λ such that the point $(x,t;\lambda)$ is transformed to the point $(\bar{x},\bar{t};\lambda)$ we have a parameter-dependent symmetry transformation. Lie theory of symmetry transformations applicable to differential equations is centered on the notion of continuous parameter λ dependent solution to solution maps and that continuous application of the symmetry transformations on the variable space-domain is invariant. By this we mean the following symbolic transformations

$$\bar{t} = \bar{t}(x,t;\lambda),\ \bar{x} = \bar{x}(x,t;\lambda);$$
$$\bar{\bar{x}} = \bar{\bar{x}}(\bar{x},\bar{t};\bar{\lambda}) = \bar{\bar{x}}(x,t;\bar{\bar{\lambda}}), \qquad (1)$$

and for some $\bar{\bar{\lambda}} = \bar{\bar{\lambda}}(\bar{\lambda},\lambda)$, the identity $\lambda = 0$ ensured that $\bar{t}(x,t;0) = t$ and $\bar{x}(x,t;0) = x$ hold for the continuous group parameter λ.

2.1 Infinitesimal generators

The groups of infinitesimal transformations of the arbitrary point (x,t) to another point $(\bar{x},\bar{t})$ according to Lie are defined by the following relations [1,4]

$$\bar{t}(x,t;\lambda) = t + \lambda\xi(x,t) + ... = t + \lambda Vt + ...;$$
$$\bar{x}(x,t;\lambda) = x + \lambda\eta(x,t) + ... = t + \lambda Vx + ..., \qquad (2)$$

where $\left.\frac{\partial \bar{t}}{\partial \lambda}\right|_{\lambda=0} = \xi(x,t)$ and $\left.\frac{\partial \bar{x}}{\partial \lambda}\right|_{\lambda=0} = \eta(x,t)$.

The operator V is given by

$$V = \xi(x,t)\partial_t + \eta(x,t)\partial_x \qquad (3)$$

where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_x = \frac{\partial}{\partial x}$. It is easy to see that (2) is the Taylor expansion of the points about $\lambda = 0$ to order one. The operator in (3) is known as the infinitesimal generators of the Lie point symmetry transformations (1), while the functions $\xi(x,t)$ and $\eta(x,t)$ are called the infinitesimals of the Lie point symmetry transformations. Once the vector field V is known, then the Lie point transformation (symmetry) can be obtained by the relation

$$(\bar{t},\bar{x}) = e^{\lambda V}(t,y),$$

where $e^{\lambda V} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} V^n$.

2.2 Flows (Lie group of symmetry transformations)

A flow or one parameter group of symmetry transformations of a space $X \subset M$ onto itself is a set of functions $f_\lambda : X \to X$ such that the following composition and identity maps are respectively defined on the space X,

(i) $f_{\lambda+\mu} = f_\lambda \circ f_\mu$;

(ii) $f_0 = id$ on X.

Theorem1. The map $f_\lambda : X \to X$ is a flow if and only if there is a vector function V on X such that $\mathbf{\bar{x}} = f_\lambda(\mathbf{x})$ is a solution of the equation

$$\frac{d\bar{\mathbf{x}}}{d\lambda}=V(\bar{\mathbf{x}}),\ \bar{\mathbf{x}}=\mathbf{x}\text{ when }\lambda=0.$$

Proof: let $f_\lambda : X \to X$ be a flow, then $f_{\lambda+\mu}(\mathbf{x}) = f_\lambda[f_\mu(\mathbf{x})]$. On differentiating this with respect to μ we have the following relation

$$\frac{d}{d\mu}f_{\lambda+\mu}(\mathbf{x}) \equiv \frac{d}{d\lambda}f_{\lambda+\mu}(\mathbf{x}) = \frac{df_\mu}{d\mu}[f_\lambda(\bar{\mathbf{x}})]. \quad (4)$$

Setting $\mu = 0$ in (4) we obtain $\frac{d\bar{\mathbf{x}}}{d\lambda} = V(\bar{\mathbf{x}})$ where $\bar{\mathbf{x}} = f_\lambda(\mathbf{x})$, $V(\mathbf{x}) = \frac{d}{d\mu}f_\mu(\mathbf{x})\Big|_{\mu=0}$.

Conversely, if

$$\frac{d\bar{\mathbf{x}}}{d\lambda}=V(\bar{\mathbf{x}}),\bar{\mathbf{x}}=\mathbf{x}\text{ when }\lambda=0 \quad (5)$$

then $\bar{\mathbf{x}} = f_\lambda(\mathbf{x}) = \mathbf{x} + \int_0^\lambda V[f_{\lambda'}(\mathbf{x})]d\lambda'$. (6)

But the function $g_\lambda(\mathbf{x}) = f_{\lambda+\mu}(\mathbf{x})$ also satisfies $g_0(\mathbf{x}) = f_\mu(\mathbf{x})$. Thus

$$f_{\lambda+\mu}(\mathbf{x}) = f_\lambda[f_\mu(\mathbf{x})]. \ \blacksquare$$

If $F : X \to \Re$ is a function then by Taylor's Theorem we have that

$$F(\bar{\mathbf{x}}) = \sum_{n=0}^{k}\frac{\lambda^{\mathbf{n}}}{n!}\frac{d^n\bar{F}}{d\lambda^n}\Bigg|_{\lambda=0} \quad (7)$$

But

$$\frac{d\bar{F}}{d\lambda} = \sum_i \frac{d\bar{x}_i}{d\lambda}\frac{\partial \bar{F}}{\partial \bar{x}_i} = \sum_i \bar{v}_i\frac{\partial \bar{F}}{\partial \bar{x}_i}\Bigg|_{\lambda=0},$$

$$= \sum_i v_i \frac{\partial F}{\partial x_i} = VF \quad (8)$$

So (7) implies

$$F(\bar{\mathbf{x}}) = \sum_{n=0}^{k}\frac{\lambda^n}{n!}V^nF = e^{\lambda V}F(x), \quad (9)$$

where $V = \sum v_i\partial_i$ is called the vector field generating the flow f_λ (commonly referred to as the symmetry generator).

2.2.1 Illustrative Examples

(1). The flow generated by the vector field $V = t\partial_x$ on $R \times R = \{(x,t) \mid x,t \in R\}$ is given by the solution to the equations

$$\frac{d\bar{x}}{d\lambda} = \bar{t}, \qquad \frac{d\bar{t}}{d\lambda} = 0 \text{ where}$$

$(\bar{x},\bar{t}) = (x,t)$ when $\lambda = 0$,

i.e. $\bar{t} = t$, $\bar{x} = x + \lambda t$. So the flow is given by

$$(\bar{x},\bar{t}) = f_\lambda(x,t) = (x+\lambda t, t). \quad (10)$$

Conversely given the flow f_λ, the vector field generating it is given by

$$\frac{d}{d\mu}f_\mu(x,t)\Bigg|_{\mu=0}\cdot\partial = t\partial_x + 0\partial_t = t\partial_x.$$

(2). The vector field $V = xt\partial_x + t^2\partial_t$ generates the flow f_λ given by the solution of the equations $\frac{d\bar{x}}{d\lambda} = \bar{x}\bar{t}$, $\frac{d\bar{t}}{d\lambda} = \bar{t}^2$. The equations respectively give the solutions

$$\bar{x} = \frac{x}{1-\lambda t} \text{ and } \bar{t} = \frac{t}{1-\lambda t}.$$

Thus

$$(\bar{x},\bar{t}) = f_\lambda(x,t) = (1-\lambda t)^{-1}(x,t). \quad (11)$$

Conversely, calculating

$$\frac{d}{d\mu}f_\mu(x,t)\Bigg|_{\mu=0}\cdot\partial = xt\partial_x + t^2\partial_t,$$

which is the vector field generating the flow.

2.3 Lie groups and Lie algebras

We present some examples of Lie groups and Lie algebras for easy appreciation of the concepts.

1). The group of rotations in two-dimensions is defined by

$$SO(2) = \{A_{2\times2} \mid A^TA = I, \det A = 1\}.$$

(12) The Lie algebra $so(2)$ of $SO(2)$ is

$$so(2) = \{L_{2\times2} \mid L^T + L = 0\}. \quad (13)$$

Now let $L = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a\varphi$

$\Rightarrow \varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have the following relations

$\varphi^2 = -I$, $\varphi^3 = -\varphi$, $\varphi^4 = \varphi^2 = -I$ etc. If $A \in SO(2)$ then we have that

$$A = e^L = I + L + \tfrac{1}{2!}L^2 + \tfrac{1}{3!}L^3\ \tfrac{1}{4!}L^4 + \ldots$$

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$$= I + a\varphi - \tfrac{1}{2!}a^2 I - \tfrac{a^3}{3!}\varphi^3 \tfrac{a^4 1}{4!} I + \ldots$$
$$= \{1 - \tfrac{1}{2!}a^2 - \tfrac{a^4 1}{4!} - \ldots\} I$$
$$+ \{a - \tfrac{1}{3!}a^3 + \tfrac{1}{5!}a^5 - \ldots\}\varphi$$
$$= \cos aI + \sin a\varphi$$

$$= \begin{pmatrix} \cos a & 0 \\ 0 & \cos a \end{pmatrix} + \begin{pmatrix} 0 & \sin a \\ -\sin a & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \qquad (14)$$

2). The group of rotations in three-dimensions is defined by

$$SO(3) = \{A_{3\times 3} \mid A^T A = I, \det A = 1\}. \qquad (15)$$

While the algebra $so(3)$ of $SO(3)$ is defined by

$$so(3) = \{L_{3\times 3} \mid L^T + L = 0\}. \qquad (16)$$

3). The group defined by

$$SO(1,1) = \{A_{2\times 2} \mid \bar{x}_2^2 - \bar{x}_1^2 = x_2^2 - x_1^2, \bar{x} = Ax, \det A = 1\}. \qquad (17)$$

While the algebra is given by

$$so(1,1) = \{L_{2\times 2} \mid e^{\lambda L} \in SO(1,1)\}. \qquad (18)$$

2.4 Lie point symmetry and nonlocal symmetry

The Lie theory of symmetry analysis of differential equations is anchored on the shore of extended (prolongation) vector fields. [4,1,2,3] For a vector field given by the relation

$$V = \xi(x,t)\partial_t + \eta(x,t)\partial_x, \qquad (19)$$

the prolongation of V to the nth order is defined by the relation

$$V^{(n)} = \xi\partial_t + \eta\partial_x + \eta'\partial_{x'} + \ldots + \eta^{(n)}\partial_{x(n)}, \qquad (20)$$

where $\eta^{(n)} = \dfrac{d^n}{dx^n}(\eta - x'\xi) + x^{(n+1)}\xi$, (21)

and $\eta^{(n)}$ is not the nth derivative of η. The action of the prolonged vector field on the differential equation is invariant is well known. Thus the general equation of order k denoted by

$$E(t, x, \dot{x}, \ldots x^k) = 0, \qquad (22)$$

is invariant under the action of the kth prolonged vector field $V^{(k)}$ given by

$$V^{(k)} E(t, x, \dot{x}, \ldots x^k)\big|_{E(t,x,\dot{x},\ldots,x^k)=0} = 0.$$

(23)

The system (23) separates into systems of partial differential equations in terms of the infinitesimals that can be solved by the method of superposition of linearly independent basis solutions $\xi_i(x,t)$ and $\eta_i(x,t)$ so that

$$V_i = \xi_i(x,t)\partial_t + \eta_i(x,t)\partial_x, \qquad (24)$$

become the infinitesimal generator of the Lie point symmetries of (22). It is well known in the literature that the totality (dimension) of (24) defined the group dimensionality of the Lie point symmetry group of (22). When (22) is of order one, the totality of (24) is infinite and there is no known algorithm of obtaining them, while the dimension is less or equal eight if it is of order two or more equation.

2.4.1 Definitions

If the infinitesimal $\xi_i(x,t)$ in (24) contains integral(s) of the dependent variable, the resulting infinitesimal generator is called nonlocal symmetry.[11,12.13]

i.e $\quad Y = \{\int \xi dt\}\partial_t + \eta\partial_x. \qquad (25)$

We note also that there are exponential nonlocal symmetries if the infinitesimal contained exponent of integral(s).[12,15] If the infinitesimals $\xi_i(x,t)$ and $\eta_i(x,t)$ in (24) are dependent on the derivative of x say $\xi(x,\dot{x},t)$ and $\eta(x,\dot{x},t)$ the resulting infinitesimal generator is called contact symmetry. Note that contact symmetries are also regarded as Lie point symmetries.

2.5 Complete symmetry groups

The concept of complete symmetry groups was generally accepted to mean the group of symmetries of differential equations which completely specify them on till recently. In this view Lie identified the symmetry groups of second-order differential equations to have not more than eight Lie point symmetries that specify them completely (any linearizable second-order differential equation has the maximum eight Lie point symmetry group).[16,17,18,19] The literature in this issue is very rich, the work of Noether on the Kepler problem could only identified five variational symmetries (also found by Lie analysis) [1,2,3] which could not specify the Kepler equation of motion. So there was a gap of not been able to obtain the complete symmetry groups for the Kepler problem in the sense of Lie. More recently, it was shown [7,20] that complete symmetry groups and algebras are not unique and the concepts of maximality and minimality of symmetry groups and algebras came to the fore. However for the purpose of this paper we intend to confine our discussion to emergence of nonlocal symmetries as by-product of the quest for complete symmetry groups of the Kepler problem for which the forerunner is Krause (1994) [we refer the interested reader to references in ref. 11,10,13], who obtained the additional three symmetries (nonlocal type) and together with the five point symmetries obtained by either Noether theorem or Lie theorem, was able to specify the equation of motion of the Kepler problem completely.

3. Exact symmetry transformations of Kepler problem in two-dimensions

We firstly review the Kepler problem to note some of its interesting properties as following. The Kepler problem has the equation of motion given by

$$\ddot{\mathbf{x}} + \frac{\mu \mathbf{x}}{r^3} = 0, \; r = |\mathbf{x}|. \tag{26}$$

The system (26) possess the angular momentum vector $\mathbf{L}$ where

$$\mathbf{L} = \mathbf{x} \wedge \dot{\mathbf{x}}. \tag{27}$$

The vector product of (26) with (27) yields the relation

$$\left(\dot{\mathbf{x}} \wedge \mathbf{L}\right)^{\cdot} + \frac{\mu \mathbf{x} \wedge \mathbf{L}}{r^3} = 0. \tag{28}$$

Using $\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$ we have that $\mathbf{x} \wedge \mathbf{L} = -r^3 \dot{\mathbf{e}}_r$, so that (28) becomes

$$\left(\dot{\mathbf{x}} \wedge \mathbf{L}\right)^{\cdot} - \mu \dot{\mathbf{e}}_r = 0, \tag{29}$$

and on integrating (29) we obtain the second conserved quantity called Laplace-Runge-Lenz (LRL) vector $\mathbf{J}$ given by

$$\left(\dot{\mathbf{x}} \wedge \mathbf{L}\right) - \mu \mathbf{e}_r = \mathbf{J} \tag{30}$$

The third conserved vector of (26) is the Hamilton's vector obtained by Hamilton in 1845; videlicet

$$\mathbf{K} = \dot{\mathbf{x}} - \frac{\mu}{L} \hat{\mathbf{L}} \wedge \mathbf{e}_r, \; L = |\mathbf{L}|. \tag{31}$$

The analysis of system (26) for its Lie point symmetries is very rich in the literature. It is well known that Lie method produced five Lie point symmetry generator which was also demonstrated by Noether method of variational symmetry theory.[10,11,13] The first five Lie point symmetry generators are given as follows

$$X_1 = \partial_t, X_2 = t\partial_t + \tfrac{2}{3} r\partial_r,$$
$$X_3 = x_2 \partial_{x_3} - x_3 \partial_{x_2}, \qquad X_4 = x_3 \partial_{x_1} - x_1 \partial_{x_3},$$
$$X_5 = x_1 \partial_{x_2} - x_2 \partial_{x_1}. \tag{32}$$

While the three additional nonlocal symmetry generators[11] are

$$Y_1 = 2\{\textstyle\int x_1 dt\}\partial_t + x_1 r\partial_r,$$
$$Y_2 = 2\{\textstyle\int x_2 dt\}\partial_t + x_2 r\partial_r,$$
$$Y_3 = 2\{\textstyle\int x_3 dt\}\partial_t + x_3 r\partial_r, \tag{33}$$

where $r^2 = x_1^2 + x_2^2 + x_3^3$. One notice that the above symmetry generators (32) and (33) separate into the following four symmetry transformations viz

i) Translation symmetries (time and special);

ii) Dilation also called self similarity or scaling symmetries (time and special);

iii) Rotation symmetries;

iv) Nonlocal symmetries.

The scaling symmetry X_2 described the Laplace-Runge-Lenz (LRL) vector of the Kepler problem which is the source of the orbit equation of (26).[13,23] However later works have established that these nonlocal symmetries are attainable by reduction of order developed by Nucci [13,21], and more also it is well known that the reduction of order process is achieved by natural reduction variables of the system via the Ermanno-Bernoulli constants [21,13,22] and as well as quasi-Ermanno-Bernoulli constants reported in ref. 14, which reduced (26) to a system of oscillator(s) and a conservation law. We note that this is applicable to most dynamical systems. The reduced system of (26) using the method of Nucci [13], Nucci and Leach [21] and the associated Lie symmetry generators are given by (34) and (35) respectively,

$$v_1'' + v_1 = 0, \quad v_2' = 0 \tag{34}$$

where $v_1 = L^2 r^{-1} - \mu$; $v_2 = r^2\dot{\theta}$ and

$$\Gamma_1 = v_2\partial_2; \Gamma_2 = \partial_\theta; \Gamma_3 = v_1\partial_1; \ \Gamma_{4\pm} = e^{\pm i\theta}\partial_1;$$
$$\Gamma_{6\pm} = e^{\pm 2i\theta}[\partial_\theta \pm i v_1 \partial_1];$$
$$\Gamma_{8\pm} = e^{\pm i\theta}[v_1\partial_\theta \pm i v_1^2 \partial_1], \tag{35}$$

where $\partial_i = \partial / \partial v_i$. Obtaining the symmetry generators of the dynamical system (26) entails the backward translation from the symmetries (35) of the reduced system (34) variables to the original variables of system (26), the scheme for doing this is available [24] and many of which are largely nonlocal symmetries in the original variables. We only list the symmetries in the original variables below:

$$\Gamma_1 = 3t\partial_t + 2r\partial_r,$$
$$\Gamma_2 = \partial_\theta,$$
$$\Gamma_3 = 2\left[\mu\int r dt - L^2 t\right]\partial_t + r\left(\mu r - L^2\right)\partial_r,$$
$$\Gamma_{4\pm} = 2\left[\int r e^{\pm i\theta} dt\right]\partial_t + r^2 e^{\pm i\theta}\partial_r,$$
$$\Gamma_{6\pm} = 2\left[\int \left(\mu r + 3L^2\right) e^{\pm 2i\theta} dt\right]\partial_t + r\left(\mu r + 3L^2\right) e^{\pm 2i\theta}\partial_r + L^2 e^{\pm 2i\theta}\partial_\theta,$$
$$\Gamma_{8\pm} = 2\left[\int \left\{2\dot{r}L^3 \pm ir\left(\mu - r^3\dot{\theta}^2\right)\left(\mu + r^3\dot{\theta}^2\right)\right\} e^{\pm i\theta} dt\right]\partial_t + r\left[2\dot{r}L^3 \pm ir\left(\mu - r^3\dot{\theta}^2\right)\left(\mu + r^3\dot{\theta}^2\right)\right]\partial_r + L^2\left(\mu - r^3\dot{\theta}^2\right) e^{\pm i\theta}\partial_\theta, \tag{36}$$

in which the factor L^2 has been included to make the expressions look simpler.

We now calculate the exact symmetry transformations of (26) from (35) as following. For the vector field $\alpha v_1 \partial_1$ where α is arbitrary constant the flow of this vector field is the function $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = \alpha \bar{v}_1 \ ; \ \frac{d\bar{v}_2}{d\lambda} = 0; \frac{d\bar{\theta}}{d\lambda} = 0. \tag{37}$$

Solving system (37) we have the following

$$\bar{v}_1 = e^{\alpha\lambda} v_1 \ ; \ \bar{v}_2 = v_2; \bar{\theta} = \theta. \tag{38}$$

The second equation in (38) implies that $\bar{L} = L$ while the first equation implies that

$$\bar{L}^2 \bar{r}^{-1} - \mu = C(L^2 r^{-1} - \mu),$$
$$\frac{r}{\bar{r}} = \mu r L^{-2} + C(1 - \mu L^{-2} r),$$
$$\bar{r} = H_1^{-1} r, \tag{39}$$

where $H_1 = \mu r L^{-2} + C(1 - \mu L^{-2} r)$, $C = e^{\alpha\lambda}$. (40)

From $\bar{r}^2 \dot{\bar{\theta}} = r^2 \dot{\theta}$ we have that

$$\frac{d\bar{t}}{dt} = H_1^{-2} \tag{41}$$

Equations (40) and (41) constitute the exact symmetry transformations of (26) with the given generator $\Gamma_3 = v_1 \partial_1$. We note that these symmetry transformations are global, that is

$$\bar{\mathbf{x}} = H_1^{-1} \mathbf{x}. \tag{42}$$

We also note that when $\mathbf{x}$ is made three-dimensional, the symmetry transformations (42) is also true. For the vector field $\alpha v_2 \partial_2$, we have the flow as $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = 0 \ ; \ \frac{d\bar{v}_2}{d\lambda} = \alpha \bar{v}_2; \frac{d\bar{\theta}}{d\lambda} = 0. \tag{43}$$

Solving system (43) we have the following

$$\bar{v}_1 = v_1 \ ; \ \bar{v}_2 = e^{\alpha\lambda} v_2; \bar{\theta} = \theta. \tag{44}$$

The second equation in (44) implies $\bar{L} = CL$ while the first equation implies that

$$\bar{L}^2 \bar{r}^{-1} - \mu = L^2 r^{-1} - \mu$$

$$\text{i.e. } C^2 L^2 \bar{r}^{-1} - \mu = L^2 r^{-1} - \mu$$

where $C = e^{\alpha\lambda}$, then

$$\frac{\bar{r}}{r} = C^2 \Rightarrow \bar{r} = C^2 r. \tag{45}$$

But $\bar{L} = CL$ implies that

$$\dot{\bar{\theta}}\bar{r}^2 = C\dot{\theta}r^2 \Rightarrow \bar{r}^2\frac{d\bar{\theta}}{d\bar{t}} = Cr^2\frac{d\theta}{dt},$$

which implies that

$$\frac{d\bar{t}}{dt} = C^3, \qquad (46)$$

i.e. $\bar{t} = d + C^3 t$,

where d is an arbitrary constant.

Consequently the exact symmetry transformations generated by the vector field $\Gamma_1 = v_2\partial_2$ for the Kepler problem is given by equations (45) and (46). If $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ denotes the Cartesian coordinates of $\mathbf{x}$ in the plane of motion then $\bar{\theta} = \theta$ implies that

$$\bar{\mathbf{x}} = C^2\mathbf{x}, \qquad (47)$$

which is the global symmetry transformation where $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ is the two dimensional Cartesian vector.

The vector field $\alpha\partial_\theta$ has the flow $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = 0\ ;\ \frac{d\bar{v}_2}{d\lambda} = 0; \frac{d\bar{\theta}}{d\lambda} = \alpha. \qquad (48)$$

Solving system (48) we have the following,

$$\bar{v}_1 = v_1\ ;\ \bar{v}_2 = v_2; \bar{\theta} = \theta + \alpha\lambda. \qquad (49)$$

Since $\bar{v}_2 = v_2 \Rightarrow \bar{L} = L$ we have that

$$\bar{L}^2\bar{r}^{-1} - \mu = L^2r^{-1} - \mu \Rightarrow \bar{r} = r, \qquad (50)$$

and $\bar{t} = t$. (51)

While the global symmetry transformations are given by $\bar{\mathbf{x}} = \mathbf{x}$ and (51). The rotation symmetry transformations are given by $\bar{\mathbf{x}} = A\mathbf{x}$, where $x_1 = r(\cos(\theta + \alpha\lambda))$,

$x_1 = r(\cos(\theta + \alpha\lambda))$; That is

$$\bar{\mathbf{x}} = \begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix} = \begin{pmatrix}\cos\alpha\lambda & -\sin\alpha\lambda\\ -\sin\alpha\lambda & \cos\alpha\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}. \qquad (52)$$

If the matrix A is arbitrary it implies that all rotation symmetry transformations are ensured.Applying the same manner of calculations we obtain the exact symmetry transformations for the vector field $\alpha e^{i\theta}\partial_\theta$,

i.e. $(\alpha_1\cos\theta + \alpha_2\sin\theta)\partial_1$

$$\bar{\mathbf{x}} = H_4^{-1}\mathbf{x}; \frac{d\bar{t}}{dt} = H_4^{-2} \qquad (53)$$

where

$$H_4 = 1 + \lambda L^{-2}\boldsymbol{\alpha}\cdot\mathbf{x},\ C = e^{\alpha\lambda},$$
$$\boldsymbol{\alpha}\cdot\mathbf{x} = \alpha_1x_1 + \alpha_2x_2\ .$$

We note here also that this is true for the case when $\mathbf{x}$ is in three-dimensions.

4. Exact symmetry transformations of Kepler problem in three-dimensions

The reduced system for (26) in three-dimensions is well known and is given by

$$u_1'' + u_1 = 0$$
$$u_2'' + u_2 = 0 \qquad (54)$$
$$u_3' = 0$$

where

$$u_1 = \left(\frac{1}{r} - L^{-2}\mu\right)\sin\theta - L^{-2}r^2\dot{r}\dot{\theta}\cos\theta,$$
$$u_1' = -L^{-2}r^2\dot{r}\dot{\phi}\sin\theta,$$
$$u_2 = L^{-1}r^2\dot{\phi}\sin\theta\cos\theta,$$
$$u_2' = -L^{-1}r^2\dot{\theta}. \qquad (55)$$

We have reported[14] that the symmetries of dynamical systems in three-dimensions can be obtained from the Lie symmetries of the reduced systems. We list here the Lie symmetry generators of the reduced system (26). They consist of sixteen generators, one viz Γ_1 for the conservation law $u_3' = 0$ and the fifteen Lie symmetry generators for the pair of harmonic oscillators (54). They are as follows

$$\Gamma_1 = u_3\partial_3,\ \Gamma_2^{jk} = u_j\partial_k,$$
$$\Gamma_3 = \partial_\phi,\ \Gamma_{4\pm}^j = e^{\pm i\phi}\partial_j,$$
$$\Gamma_{5\pm} = e^{\pm 2i\phi}(\partial_\phi + i\mathbf{u}\cdot\partial),$$
$$\Gamma_{6\pm}^j = e^{\pm i\phi}u_j(\partial_\phi + i\mathbf{u}\cdot\partial) \qquad (56)$$

where $j,k = 1,2; \partial_j = \partial/\partial u_j$ and $\mathbf{u}\cdot\partial = u_1\partial_1 + u_2\partial_2$. The symmetry representations of (56) in the original variables are very much complicated than that of section 2 above. We now compute the symmetry transformation generated by the vector field

$\alpha\Gamma_2^{11} = \alpha u_1\partial_1$ for the Kepler problem. The symmetry transformation generated by this vector field is the transformation f given by

$(\bar{u}_j, \bar{\phi}) = f(u_j, \phi)$ where

$$\bar{u}_1 = Cu_1, \bar{u}_2 = u_2, \bar{u}_3 = u_3, \bar{\phi} = \phi, C = e^{\alpha\lambda} \quad (57)$$

from which it follows that

$$\bar{u}_1' = Cu_1', \bar{u}_2' = u_2', \bar{L} = L. \quad (58)$$

The relations (41) in ref. 14, imply that

$$u_2^2 \sec^2\theta + (u_2')^2 = 1.$$

Thus from the invariance of u_2 and u_2' in (56) and (58) we note that $\sec\bar{\theta} = \sec\theta$,

$$\text{i.e. } \bar{\theta} = \theta. \quad (59)$$

The relations in (55) imply that

$$u_1 = \left(\frac{1}{r} - \mu L^{-2}\right)\sin\theta - u_1'\theta'\cot\theta. \quad (60)$$

Since L, θ' and $\cot\theta$ are invariants of this transformation, the first relation in (56) becomes

$$\left(\frac{1}{\bar{r}} - \mu L^{-2}\right)\sin\theta - \bar{u}_1'\theta'\cot\theta = C\left(\frac{1}{r} - \mu L^{-2}\right)\sin\theta - Cu_1'\theta'\cot\theta, \quad (61)$$

which reduces to

$$\left(\frac{1}{\bar{r}} - \mu L^{-2}\right) = C\left(\frac{1}{r} - \mu L^{-2}\right); \quad (62)$$

$$\text{i.e. } \bar{r} = H_2^{-1} r,$$

where $H_2 = \mu L^{-2} r + C(1 - \mu L^{-2} r)$.

The relation $\bar{u}_2' = u_2'$ in (58) implies that

$$\bar{L}^{-1}\bar{r}^2\dot{\bar{\theta}} = L^{-1}r^2\dot{\theta}$$

$$\text{i.e. } \frac{d\bar{t}}{dt} = \left(\frac{\bar{r}}{r}\right)^2 = H_2^{-2} \quad (63)$$

In view of equations (62), (63) and the relations $\bar{\theta} = \theta$, $\bar{\phi} = \phi$ in (57) and (58) the required exact symmetry transformation of the Kepler problem in three-dimensions for the vector field $\alpha u_1\partial_1$ is given by

$$\bar{\mathbf{x}} = H_2^{-1}\mathbf{x}; \ \frac{d\bar{t}}{dt} = H_2^{-2}. \quad (64)$$

Thus, we hereby depict in the following table some of the vector fields with their corresponding exact symmetry transformations below:

Table 1 – Vector fields and exact symmetry transformations

Vector fields	Exact symmetry transformations
$\alpha\Gamma_2^{22} = \alpha u_2\partial_2$	$\bar{r} = Cr$, $\bar{t} = d + C^2 t$ (d is constant) and $\bar{\mathbf{x}} = C\mathbf{x}$ is the Global exact symmetry.
$\alpha\Gamma_2^{12} = \alpha u_1\partial_2$	$\bar{r} = H^{-1}r$, $\frac{d\bar{t}}{dt} = H^{-2}$, where $H = \left[1 + \lambda r^{-1}\{(L^2 r^{-1} - \mu)\tan\theta\cos\theta - \dot{r}L\sin\theta\}\right]^{-1}$.
$\alpha\Gamma_2^{21} = \alpha u_2\partial_1$	$\bar{r} = H^{-1}r$, $\frac{d\bar{t}}{dt} = H^{-2}$, where $H = \left[1 + \lambda r L^{-2}\cos ec^2\theta\cos\theta(\alpha - \theta\phi^{-1}\cot\theta)\right]$.
$\alpha\Gamma_3 = \alpha u_3\partial_3$	$\bar{\mathbf{x}} = \mathbf{x}$, $\bar{t} = d + C^{-1}t$, where d is a constant
$\alpha\Gamma_4^2 = \alpha e^{i\phi}\partial_2$ Note that the sine part can be easily deduced from this obviously.	$\bar{r} = H^{-1}r$, $\frac{d\bar{t}}{dt} = H^{-2}$, where $H = \left[1 + \lambda\alpha r^{-1}\cos\phi\tan\theta\right]^{-1}$.
$\alpha\Gamma_4^1 = \alpha e^{i\phi}\partial_1$ The sine part is deductive from this cosine part obviously.	$\bar{r} = H^{-1}r$, $\frac{d\bar{t}}{dt} = H^{-2}$ where $H = \left[1 + \lambda r L^{-2}\cos ec\theta\cos\phi(\alpha - \theta\phi^{-1}\cot\theta)\right]^{-1}$,

$\alpha\Gamma_3 = \alpha\partial_\phi$	$\bar{r} = r, \bar{t} = t$.

We note that the rotation symmetries of the Kepler problem in three-dimension are obtainable from the vector field $\alpha\Gamma_3 = \alpha\partial_\phi$ which has the rotation symmetries denoted by $\bar{\mathbf{x}} = B\mathbf{x}$, B is a scalar 3x3-matrix. That is by setting $\mathbf{x} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ as the spherical coordinates of the motion we have the rotation symmetry transformation about the z-axis as

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha\lambda & -\sin\alpha\lambda & 0 \\ \sin\alpha\lambda & \cos\alpha\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (65)$$

If B is an arbitrary rotation matrix then the rotation symmetry is globally defined.

We report that the exact symmetries of the remaining six vector fields are computable as well following the same method diligently. We note also that the Hamilton vector $\mathbf{K}$ for the Kepler problem is given by[25] $\mathbf{K} = \dot{\mathbf{x}} - \mu L^{-2} r^{-1}(\mathbf{L} \wedge \mathbf{x})$ (This is a constant multiple of the expression for $\mathbf{K}$ given in ref.14). This expression for $\mathbf{K}$ yields the relation[14]

$$K_\pm = K_1 \pm iK_2 = (\upsilon_1 \pm i\upsilon_1')e^{\pm i\phi},$$

where

$$\upsilon_1 = \dot{r}\sin\theta + r(1 - \mu L^{-2} r)\cos\theta\dot{\theta},$$
$$\upsilon_1' = (1 - \mu L^{-2} r)\sin\theta\dot{\phi}. \quad (66)$$

We note that one could consider instead of (54), the same system of equations with u_1 replaced with υ_1, and its Lie symmetries to obtain the exact symmetries of the original system.[14]

5. Concluding remarks

We have demonstrated that the exact (actual) symmetry transformations of the Kepler problem can be calculated from the symmetries of its reduced systems rather than just obtaining the symmetry generators (vector fields) that are often complicated in their representations as they are in their nonlocal symmetry forms (37). Hitherto the exact symmetry transformations computation as demonstrated above is new. We report here that we have devised and utilized this computational method to obtain the exact symmetries of other dynamical systems that are reducible to systems of oscillator(s) and conservation law(s). Consequently the complicated nonlocal symmetry representations of dynamical systems are simply realizable in their simple explicit forms as shown using the Kepler problem as a vehicle. In our recent works the Kepler problem with drag, the generalized Kepler problem, the MICZ problem and a host of other dynamical systems with complicated nonlocal symmetries have proven to admit this computational method for obtaining their exact symmetries in both two- and three-dimensions. These are subject for further discussions.

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