Finite Difference Schemes of Unsteady Boundary Layer Flow and Heat Transfer over a Stretching Surface in a Micropolar Fluid

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Abstract: - In this paper, several numerical schemes are presented for the problem of unsteady boundary layer flow and heat transfer of an incompressible micropolar fluid over a stretching sheet when the sheet is stretched in its own plane. We consider the equations given in [1] and then we introduce several numerical schemes. The velocity and temperature are assumed to vary linearly with the distance along the sheet. Two equal and opposite forces are impulsively applied along the x-axis so that the sheet is stretched, keeping the origin fixed in a micropolar fluid.

Key-Words: - Finite Differences, Numerical Schemes, Heat transfer, Micropolar fluid

1 Introduction

In [1], the authors have considered the flow of an incompressible micropolar fluid in the region \( y > 0 \) driven by a plane surface located at \( y = 0 \) with a fixed end at \( x = 0 \). It is assumed that the surface is stretched in the x-direction such that the temperature and x-component of the velocity vary linearly along it, i.e. \( T_s(x) = T_0 + ax \) and \( u_s(x) = cx \) respectively, where \( a \) and \( c \) are arbitrary positive constants. The simplified two-dimensional equations governing the flow in the boundary layer of a steady, laminar and incompressible micropolar fluid are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \frac{\mu + \kappa}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\kappa}{\rho} \frac{\partial N}{\partial y} \tag{2}
\]

\[
\rho \left( \frac{\partial N}{\partial t} + u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} \right) = \gamma \frac{\partial^2 N}{\partial y^2} - \kappa \left( 2N + \frac{\partial u}{\partial y} \right) \tag{3}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2} \tag{4}
\]

where the initial conditions are written again here in a more general and compact form

**Initial Conditions:**

The following functions of \( x,y,t \) are known (given) for \( t=0 \), i.e.

\[
v(x,y,0) = v_0(x,y) \]

\[
u(x,y,0) = u_0(x,y) \]
\[ N(x,y,0) = N_0(x,y) \]
\[ T(x,y,0) = T_0(x,y) \]
Also \[ v(x,0,t) = v^*(x,t) \]

Here \( u \) and \( v \) are the velocity components along the \( x \)- and \( y \)-axes, respectively, \( t \) is time, \( N \) is the microrotation or angular velocity whose direction of rotation is in the \( x-y \) plane, \( T \) is temperature, \( \mu \) is dynamic viscosity, \( \rho \) is the density, \( j \) is microinertia per unit mass, \( \gamma \) is spin gradient viscosity, \( \kappa \) is the vortex viscosity and \( \alpha \) is thermal diffusivity.

Further, \( n \) is a constant where \( 0 \leq n \leq 1 \). The case \( n = 0 \), which indicates \( N = 0 \) at the wall represents concentrated particle flows in which the microelements close to the wall surface are unable to rotate (Jena and Mathur [11]). This case is also known as the strong concentration of microelements (Guram and Smith [12]). The case \( n = \frac{1}{2} \) indicates the vanishing of antisymmetric part of the stress tensor and denotes weak concentration of microelements (Ahmadi [13]). The case \( n = 1 \) as suggested by Peddieson [14] is used for the modeling of turbulent boundary layer flows. In this paper we consider only the case of \( n = \frac{1}{2} \).

In this paper, we attempt to solve numerically the systems of (1), (2), (3), (4) with the aforementioned initial conditions. Note that a simple numerical scheme is given also in [26]. In [27], several numerical schemes are given for the equations

\[ \frac{\partial u}{\partial t} = \text{div}(\nabla u) \]
\[ \frac{\partial v}{\partial t} = \text{div}(\nabla v) + \lambda u' \]
\[ \frac{\partial u}{\partial t} = \text{div}(\nabla u') + \lambda u'' \]

Accompanied by appropriate initial mconditions. Some other relevant studies can be found in [28], [29], [30].

2 Numerical Schemes for the system of (1), (2), (3) and (4)

The discretization in (1), (2), (3), (4) renders to \( u_{j,k}^i, v_{j,k}^i, N_{j,k}^i, T_{j,k}^i \) where \( i \) represents the (discrete) time and \( j,k \) the spatial coordinates. The steps of the discretization are \( h, h_1, h_2 \) with respect to \( t, x, y \). So, we have introduced a grid of points in the \( \mathbb{R}^3 \) space of \((t,x,y)\)

Here we are interested only for Explicit Numerical Schemes (because of the initial conditions), while in a boundary value problem we could use also implicit numerical schemes.

Then as a first explicit numerical scheme we can propose considering the equations (2), (3), (4) the following

\[
\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h} = -u_{j,k}^i \frac{u_{j,k}^{i+1} - u_{j-1,k}^i}{h_1} - v_{j,k}^i \frac{u_{j,k}^{i+1} - u_{j,k-1}^i}{h_2} + \frac{\mu + \kappa}{\rho} \frac{u_{j,k}^{i+1} - u_{j,k-1}^i}{h_2^2} + \frac{\kappa N_{j,k}^i - N_{j,k-1}^i}{\rho h_2} + \frac{1}{\rho j} \left( \frac{N_{j,k}^i - 2N_{j,k-1}^i + N_{j,k-2}^i}{h_2^2} - \kappa \left( \frac{N_{j,k}^i + u_{j,k}^{i+1} - u_{j,k-1}^i}{h_2} \right) \right)
\]

\[
\frac{T_{j,k}^{i+1} - T_{j,k}^i}{h} = -u_{j,k}^i \frac{T_{j,k}^{i+1} - T_{j-1,k}^i}{h_1} - v_{j,k}^i \frac{T_{j,k}^{i+1} - T_{j,k-1}^i}{h_2} + \frac{a}{h_2^2} \left( T_{j,k}^{i+1} - 2T_{j,k-1}^i + T_{j,k-2}^i \right)
\]

Therefore one can compute \( u_{j,k}^{i+1}, N_{j,k}^i, T_{j,k}^i \) from \( u_{j,k}^i, N_{j,k}^i, T_{j,k}^i \).

The point is how to compute \( v_{j,k}^{i+1} \).

To this end we consider (1) at the point \( i+1 \)

\[
\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h_1} = \frac{v_{j,k}^{i+1} - v_{j,k}^{i+1}}{h_2}
\]

So the differences \( v_{j,k}^{i+1} - v_{j,k}^{i+1} \) can be computed and \( v_{j,k}^{i+1} \) starting from \( v_{j,0} \).

because \( v_{j,0} = v^*(j h_1, i h) \) due to

\[ v(x,0,t) = v^*(x,t) \]
see the last initial condition

Numerical examples and numerical experiments are in accordance with [1] and [26].
Furthermore as a second explicit numerical scheme we can propose considering the equations (2), (3), (4) the following

$$\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h} = -u_{j,k}^i \frac{u_{j+1,k}^i - u_{j-1,k}^i}{h} - v_{j,k}^i \frac{u_{j,k+1}^i - u_{j,k-1}^i}{h} +$$

$$+ \left( \frac{\mu + \kappa}{\rho} \right) \frac{u_{j+1,k}^i - 2u_{j,k}^i + u_{j-1,k}^i}{h^2} + \frac{\kappa N_{j,k+1}^i - N_{j,k}^i}{\rho h_2}$$

$$\frac{N_{j,k+1}^i - N_{j,k}^i}{h} = -u_{j,k}^i \frac{N_{j+1,k}^i - N_{j,k}^i}{h_1} - v_{j,k}^i \frac{N_{j,k+1}^i - N_{j,k-1}^i}{h_2} +$$

$$+ \frac{1}{\rho j} \left( \frac{N_{j,k+1}^i - 2N_{j,k}^i + N_{j,k-1}^i}{h_2^2} - \kappa \left( 2N_{j,k}^i + \frac{u_{j,k}^i - u_{j,k-1}^i}{h_2} \right) \right)$$

$$T_{j,k+1}^i - T_{j,k}^i = -u_{j,k}^i \frac{T_{j+1,k}^i - T_{j-1,k}^i}{h} - v_{j,k}^i \frac{T_{j,k+1}^i - T_{j,k-1}^i}{h} +$$

$$+ \left( \gamma N_{j,k+1}^i - 2N_{j,k}^i + N_{j,k-1}^i \right) \frac{h^2}{h_2} - \kappa \left( 2N_{j,k}^i + \frac{u_{j,k}^i - u_{j,k-1}^i}{h_2} \right)$$

Also one can compute $u_{j,k}^{i+1}, N_{j,k}^{i+1}, T_{j,k}^{i+1}$ from $u_{j,k}^{i}, N_{j,k}^{i}, T_{j,k}^{i}$ computing again the $v_{j,k}^{i+1}$ from

$$\frac{u_{j,k}^{i+1} - u_{j,k}^{i}}{h} = -u_{j,k}^i \frac{u_{j+1,k}^i - u_{j-1,k}^i}{h} - v_{j,k}^i \frac{u_{j,k+1}^i - u_{j,k-1}^i}{h} +$$

(The differences $v_{j,k}^{i+1} - v_{j,k}^{i}$ can be computed and $v_{j,k}^i$ starting from $v_{j,0}^i$, because $v_{j,0}^i = v^*(jh_i,ih)$ due to $v(x,0,t) = v^*(x,t)$)

As a third explicit numerical scheme

$$\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h} = \frac{1}{2} \left( -u_{j,k}^i \frac{u_{j+1,k}^i - u_{j-1,k}^i}{h} - v_{j,k}^i \frac{u_{j,k+1}^i - u_{j,k-1}^i}{h} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{u_{j+1,k}^i - 2u_{j,k}^i + u_{j-1,k}^i}{h^2} \right) + \frac{\kappa N_{j,k+1}^i - N_{j,k}^i}{\rho h_2}$$

$$\frac{N_{j,k+1}^i - N_{j,k}^i}{h} = \frac{1}{2} \left( -u_{j,k}^i \frac{N_{j+1,k}^i - N_{j,k}^i}{h_1} - v_{j,k}^i \frac{N_{j,k+1}^i - N_{j,k-1}^i}{h_2} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{N_{j+1,k}^i - 2N_{j,k}^i + N_{j,k-1}^i}{h_2^2} \right) + \frac{\kappa N_{j,k+1}^i - N_{j,k}^i}{\rho h_2}$$

$$T_{j,k+1}^i - T_{j,k}^i = \frac{1}{2} \left( -u_{j,k}^i \frac{T_{j+1,k}^i - T_{j-1,k}^i}{h} - v_{j,k}^i \frac{T_{j,k+1}^i - T_{j,k-1}^i}{h} + \left( \gamma N_{j,k+1}^i - 2N_{j,k}^i + N_{j,k-1}^i \right) \frac{h^2}{h_2} \right) + \kappa \left( 2N_{j,k}^i + \frac{u_{j,k}^i - u_{j,k-1}^i}{h_2} \right)$$

Clearly the third explicit numerical scheme is produced by considering the averages of the right-hand sides of the equations of the first and the second explicit numerical scheme.

As in the two first explicit numerical schemes we can compute $u_{j,k}^{i+1}, N_{j,k}^{i+1}, T_{j,k}^{i+1}$ from $u_{j,k}^{i}, N_{j,k}^{i}, T_{j,k}^{i}$ computing again the $v_{j,k}^{i+1}$
3 Advanced Numerical Schemes for
the system of (1), (2), (3) and (4)

Using the methodologies of [31] we can also have the following interesting numerical schemes:

**fourth explicit numerical scheme:**

$$\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h} = -u_{j-1,k}^i u_{j,k+1}^i - v_{j,k}^i \frac{u_{j,k}^i - u_{j,k-1}^i}{h} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{u_{j,k}^i - 2u_{j,k+1}^i + u_{j,k-1}^i}{h^2} + \frac{\nu_{j,k}^i - \nu_{j,k-1}^i}{h^2} + + \left( \frac{\mu + \kappa}{\rho} \right) \left( u_{j,k+1}^i - 2u_{j,k}^i + u_{j,k-1}^i \right) + \frac{\kappa N_{j,k}^i - N_{j,k-1}^i}{h^2}
$$

$$\frac{N_{j,k}^i - N_{j,k-1}^i}{h} = -u_{j-1}^i \frac{N_{j,k}^i - N_{j,k-1}^i}{h} + \frac{1}{\rho j} \left( \gamma N_{j,k}^i - 2N_{j,k+1}^i - N_{j,k-1}^i + \kappa \left( 2N_{j,k}^i + \frac{u_{j,k}^i - u_{j,k-1}^i}{h^2} \right) \right)
$$

$$\frac{T_{j,k}^i - T_{j,k}^i}{h} = -u_{j-1}^i \frac{T_{j,k}^i - T_{j,k}^i}{h} + \frac{1}{\rho j} \left( \gamma T_{j,k}^i - 2T_{j,k+1}^i + T_{j,k-1}^i \right) + \frac{T_{j,k}^i - T_{j,k}^i}{h^2} + \frac{T_{j,k}^i - T_{j,k}^i}{h^2}
$$

So one can compute $u_{j,k}^{i+1}, N_{j,k}^i, T_{j,k}^i$ from $u_{j,k}^i, N_{j,k}^i, T_{j,k}^i$

and also

$$\frac{u_{j,k}^{i+1} - u_{j-1,k}^i}{h} = -u_{j-1,k}^i \frac{u_{j,k+1}^i - u_{j,k-1}^i}{h} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{u_{j,k}^i - 2u_{j,k+1}^i + u_{j,k-1}^i}{h^2} + \frac{\kappa N_{j,k}^i - N_{j,k-1}^i}{h^2}
$$

Quite similarly we can have a **fifth explicit numerical scheme:**

$$\frac{u_{j,k}^{i+1} - u_{j,k}^i}{h} = -u_{j-1,k}^i \frac{u_{j,k+1}^i - u_{j,k-1}^i}{h} + \left( \frac{\mu + \kappa}{\rho} \right) \left( u_{j,k+1}^i - 2u_{j,k}^i + u_{j,k-1}^i \right) + \frac{\kappa N_{j,k}^i - N_{j,k-1}^i}{h^2}
$$

$$\frac{N_{j,k}^i - N_{j,k-1}^i}{h} = -u_{j-1}^i \frac{N_{j,k}^i - N_{j,k-1}^i}{h} + \frac{1}{\rho j} \left( \gamma N_{j,k}^i - 2N_{j,k+1}^i - N_{j,k-1}^i + \kappa \left( 2N_{j,k}^i + \frac{u_{j,k}^i - u_{j,k-1}^i}{h^2} \right) \right)
$$

$$\frac{T_{j,k}^i - T_{j,k}^i}{h} = -u_{j-1}^i \frac{T_{j,k}^i - T_{j,k}^i}{h} + \frac{1}{\rho j} \left( \gamma T_{j,k}^i - 2T_{j,k+1}^i + T_{j,k-1}^i \right) + \frac{T_{j,k}^i - T_{j,k}^i}{h^2} + \frac{T_{j,k}^i - T_{j,k}^i}{h^2}
$$

A sixth numerical scheme can be constructed by fourth and fifth considering the averages of the right hand sides of the equations in a complete analogy with the way of construction of the third numerical scheme from the first and second ones.

Several other numerical schemes can be constructed if for example substitute the second order derivatives like

$$\frac{u_{j,k}^i - 2u_{j,k-1}^i + 2u_{j,k-2}^i}{h^2}$$

with

$$\frac{1}{3} \left( \frac{u_{j,k}^i - 2u_{j,k-1}^i + u_{j,k-2}^i}{h^2} + \frac{u_{j,k+1}^i - 2u_{j,k}^i + u_{j,k-1}^i}{h^2} + \frac{u_{j,k+1}^i - 2u_{j,k+1}^i + u_{j,k}^i}{h^2} \right)$$
\[ N_{j,k}^i - 2N_{j,k-1}^i + N_{j,k-2}^i \]
\[ \frac{1}{h_2^2} \]

with
\[ \frac{1}{3} \left( \frac{N_{j,k}^i - 2N_{j,k-1}^i + N_{j,k-2}^i}{h_2^2} + \frac{N_{j,k+1}^i - 2N_{j,k}^i + N_{j,k-1}^i}{h_2^2} \frac{N_{j,k+1}^i - 2N_{j,k+1}^i + N_{j,k}^i}{h_2^2} \right) \]

\[ T_{j,k}^i - 2T_{j,k-1}^i + T_{j,k-2}^i \]
\[ \frac{1}{h_2^2} \]

with
\[ \frac{1}{3} \left( \frac{T_{j,k}^i - 2T_{j,k-1}^i + T_{j,k-2}^i}{h_2^2} + \frac{T_{j,k+1}^i - 2T_{j,k}^i + T_{j,k-1}^i}{h_2^2} \frac{T_{j,k+1}^i - 2T_{j,k+1}^i + T_{j,k}^i}{h_2^2} \right) \]

without exhausting the variety at this point.

4 Conclusion

In this paper, several numerical schemes for the solution of the unsteady Boundary Layer Flow and Heat Transfer over a Stretching Surface in a Micropolar Fluid have been proposed. The investigated problem appear in the mathematical modeling of problems in fluid mechanics.

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