A Parallel Finite Difference Method For Fourth Order Parabolic Equations

Qinghua Feng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
fqhua@sina.com

Bin Zheng
Shandong University of Technology
School of Science
Zhangzhou Road 12, Zibo, 255049
China
zhengbin2601@126.com

Abstract: In this paper, we present a high order implicit scheme for fourth order parabolic equations. Then we derive eight saul’yev asymmetry schemes, based on which and the concept of domain splitting we construct a class of alternating group explicit method with intrinsic parallelism. The method is verified to be unconditionally stable and convergent. In order to verify the accurate of the method we give a numerical example at the end of the paper.

Key–Words: parabolic equations, alternating group, parallel computation, unconditionally stable, finite difference

1 Introduction

Finite difference method is one of the most frequently used numerical methods. As we all know, Most of explicit methods are short in stability and accuracy, while implicit methods usually have good stability, but are complex in computing, and need to solve large equation set in the cost of large memory spaces and CPU cycles. Thus it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. We notice that a so-called AGE method based on the concept of domain decomposition is widely cared for its intrinsic parallelism and absolute stability, which was originally presented for solving diffusion equations in [1] by Evans. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and the computation in the whole domain can be divided into many sub-domains, Then the numerical solutions at each group can be obtained independently, which highly cuts down the running computing time. So the AGE method is of obvious parallelism. Furthermore, by alternating use of asymmetry schemes at adjacent grid points and different time levels, the AGE method can lead to counteraction of truncation error, and then increase the accurate of numerical solution. The AGE method was soon applied to two point boundary value problems. Based on the concept of AGE method, a class of domain splitting method was presented in [6-7]. The developed methods have the same advantages of parallelism and absolute stability as the AGE method in [1], but we notice that almost all the methods have no more than four order accuracy for spatial step. To our knowledge researches on alternating group explicit method for fourth order parabolic equations have been scarcely presented.

In this paper we consider the following periodic boundary value problem of fourth order parabolic equations:

\[
\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, \quad -\infty < x < \infty, \quad 0 \leq t \leq T
\]

\[
u(x, 0) = f(x), \quad u(x + l, t) = u(x, t)\]  

In section 2, we present an \(O(\tau^2 + h^6)\) order unconditionally stable symmetry implicit scheme, and construct an AGE method based an the scheme. Stability analysis and convergence analysis are given in section 3. In section 4, results of numerical experiment are presented.

2 The AGE Method

The domain \(\Omega : [0, l] \times [0, T]\) will be divided into \((m \times \xi)\) meshes with spatial step size \(h = \frac{l}{m}\) in x direction and the time step size \(\tau = \frac{T}{\xi}\). Grid points are denoted by \((x_i, t_n), x_i = ih(i = 0, 1, \cdots, m), t_n = nt(n = 0, 1, \cdots, \frac{T}{\tau})\). The numerical solution of (1) is denoted by \(u^n_i\), while the exact solution \(u(x_i, t_n)\).

Let \(\delta_1 u^n_i = \frac{u^{n+1}_i - u^n_i}{\tau}\), \(\delta_2 u^n_i = \frac{u^n_{i+1} - u^n_i}{h}\).
\[ u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n \] 
\[ \frac{1}{h^4} \] 

We present an implicit finite difference scheme with parameters for solving (1) as below:

\[
\lambda_2 \delta_t u_{i+2}^n + \lambda_1 \delta_t u_{i+1}^n + \lambda_0 \delta_t u_i^n + \lambda_1 \delta_t u_{i-1}^n + \lambda_2 \delta_t u_{i-2}^n = \frac{1}{2} (\delta_x^2 u_{i+1}^n + \delta_x^2 u_i^n) \quad (2)
\]

It follows that

\[
\lambda_2 (\frac{\partial u}{\partial t})_{i+2}^n + \lambda_1 (\frac{\partial u}{\partial t})_{i+1}^n + \lambda_0 (\frac{\partial u}{\partial t})_i^n + \lambda_1 (\frac{\partial u}{\partial t})_{i-1}^n + \lambda_2 (\frac{\partial u}{\partial t})_{i-2}^n \\
+ \lambda_2 (\frac{\partial u}{\partial t})_{i+2}^n + \frac{7}{2} \lambda_2 (\frac{\partial^2 u}{\partial t^2})_{i+2}^n + \lambda_1 (\frac{\partial^2 u}{\partial t^2})_{i+1}^n + \lambda_2 (\frac{\partial^2 u}{\partial t^2})_{i+2}^n
\]

\[
= \frac{\partial^4 u}{\partial x^4} + \frac{h^2}{6} \frac{\partial^6 u}{\partial x^6} + \frac{504 h^4}{8!} \left( \frac{\partial^6 u}{\partial x^6} \right)_i^n + \frac{7}{2} \left( \frac{\partial^5 u}{\partial x^5} \right)_i^n + \frac{60 \tau h^2}{720} \left( \frac{\partial^7 u}{\partial x^7} \right)_i^n + 252 \tau h^4 \left( \frac{\partial^9 u}{\partial x^9} \right)_i^n = O(\tau^2 + h^6)
\]

Considering \( \frac{\partial^k u}{\partial t^k} = (-1)^k \frac{\partial^k u}{\partial x^k} \), we have

\[
[-\lambda_2 - \lambda_1 - \lambda_0 - \lambda_1 - 1](\frac{\partial^4 u}{\partial x^4})_i^n \\
+ [2\lambda_2 + \lambda_1 - \lambda_1 - 2\lambda_2] h(\frac{\partial^5 u}{\partial x^5})_i^n \\
+ [-2\lambda_2 - \frac{1}{2} \lambda_1 - \frac{1}{2} \lambda_1 - 2\lambda_2 + \frac{1}{6} h^2] \frac{\partial^6 u}{\partial x^6} )_i^n \\
+ [\frac{8}{3} \lambda_2 + \frac{1}{3} \lambda_1 - 1 - \frac{3}{2} \lambda_1 - \frac{8}{3} \lambda_2 h^3] \frac{\partial^7 u}{\partial x^7} )_i^n \\
+ [-\frac{24}{41} \lambda_2 - \frac{1}{4} \lambda_1 - \frac{1}{4} \lambda_1 - \frac{24}{41} h^4] \frac{\partial^8 u}{\partial x^8} )_i^n \\
+ [\frac{1}{2} \lambda_2 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_2 - \frac{1}{2} \lambda_2] (\frac{\partial^9 u}{\partial x^9})_i^n \\
+ [-\lambda_2 - \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_2] [\tau h(\frac{\partial^9 u}{\partial x^9})_i^n \\
+ [\lambda_2 + \frac{1}{4} \lambda_1 + 1 - \frac{1}{4} \lambda_1 - \lambda_2 - \frac{60}{720} \tau h^3] \frac{\partial^{10} u}{\partial x^{10}} )_i^n \\
+ [-\frac{4}{3} \lambda_2 - \frac{1}{3} \lambda_1 + \frac{1}{3} \lambda_1 + \frac{4}{3} \lambda_2] [\tau h^3 (\frac{\partial^{11} u}{\partial x^{11}} )_i^n \\
+ [\frac{8}{41} \lambda_2 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{8}{41} \lambda_2 - \frac{252}{81} \tau h^4] \frac{\partial^{12} u}{\partial x^{12}} )_i^n
\]

\[ = O(\tau^2 + h^6) \]

Let

\[
-\lambda_2 - \lambda_1 - \lambda_0 - \lambda_1 - 1 = 0 \\
2\lambda_2 + \lambda_1 - \lambda_1 - 2\lambda_2 = 0 \\
-2\lambda_2 - \frac{1}{2} \lambda_1 - \frac{1}{2} \lambda_1 - 2\lambda_2 + \frac{1}{6} = 0 \\
\frac{8}{3} \lambda_2 + \frac{1}{3} \lambda_1 - \frac{1}{3} \lambda_1 - \frac{8}{3} \lambda_2 = 0 \\
-\frac{24}{41} \lambda_2 - \frac{1}{3} \lambda_1 - \frac{1}{3} \lambda_1 - \frac{24}{41} \lambda_2 = 0 \\
\lambda_2 + \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 - \lambda_2 = 0 \\
\lambda_2 + \frac{1}{3} \lambda_1 + \frac{1}{3} \lambda_1 - \frac{8}{3} \lambda_2 = 0 \\
\frac{8}{41} \lambda_2 - \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 + \frac{8}{41} \lambda_2 = 0 \\
\lambda_2 - \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 - \frac{1}{2} \lambda_1 = \frac{124}{720} = 0 \\
\lambda_1 = \frac{124}{720} = 0 \\
\lambda_0 = \frac{474}{720}
\]

We denote (2) as

\[
-\frac{1}{720} \delta_t u_{i+2}^n + \frac{124}{720} \delta_t u_{i+1}^n + \frac{474}{720} \delta_t u_i^n + \frac{124}{720} \delta_t u_{i-1}^n + \frac{124}{720} \delta_t u_{i-2}^n
\]

and obviously the truncation error of (3) is \( O(\tau^2 + h^6) \).

Let \( U^n = (u_1^n, u_2^n, \cdots, u_m^n) \), \( r = \frac{360}{h^2} \), then from (3) we have

\[
(-1 + r) u_{i+2}^{n+1} + (124 - 4r) u_{i+1}^{n+1} + (474 + 6r) u_i^{n+1} + (124 - 4r) u_{i-1}^{n+1} + (-1 - r) u_{i-2}^{n+1} + (124 + 4r) u_{i+1}^{n+1} + (474 - 6r) u_i^{n+1} + (124 + 4r) u_{i-1}^{n+1} + (-1 - r) u_{i-2}^{n+1}
\]

(4)

Based on (4), we present eight sawt fev asymmetry schemes to solve \( u_i^{n+1} \):

\[
(474 + 3r) u_{i+1}^{n+1} + (248 - 4r) u_{i+1}^{n+1} + (-2 + r) u_{i+2}^{n+1} = -2ru_{i-2}^{n+1} + 8ru_{i-1}^{n+1} + (474 - 9r) u_i^{n+1} + (248 + 4r) u_{i+1}^{n+1} + (-2 - r) u_{i+2}^{n+1}
\]

(5)

\[
(124 - 3r) u_{i}^{n+1} + (474 + 3r) u_{i}^{n+1} + (124 - 3r) u_{i+1}^{n+1}
\]

\[
+ (-2 + r) u_{i+2}^{n+1} = -2ru_{i-2}^{n+1} + (124 + 7r) u_i^{n+1}
\]

\[
+ (474 - 9r) u_i^{n+1} + (124 + 5r) u_{i+1}^{n+1} + (-2 - r) u_{i+2}^{n+1}
\]

(6)

\[
(-2 + r) u_{i}^{n+1} + (124 - 5r) u_{i}^{n+1} + (474 + 9r) u_{i+1}^{n+1}
\]

\[
+ (124 - 7r) u_{i+1}^{n+1} + 2ru_{i+2}^{n+1} = (-2 - r) u_{i+2}^{n+1}
\]
\[(+124 + 3r)u_{i-1}^n + (474 - 3r)u_i^n + (124 + r)u_{i+1}^n) \tag{7}\]
\[(-2 + r)u_{i-2}^{n+1} + (248 - 4r)u_{i-1}^{n+1} + (474 + 9r)u_i^{n+1}\]
\[-8ru_{i-1}^{n+1} + 2ru_{i+2}^{n+1} = (-2 - r)u_i^n\]
\[+ (248 + 4r)u_{i-1}^n + (474 - 3r)u_i^n\]
\[2ru_{i-2}^{n+1} - 8ru_{i+1}^{n+1} + (474 + 9r)u_i^{n+1}\]
\[+(248 - 4r)u_{i+1}^{n+1} + (-2 + r)u_{i+2}^{n+1} = (474 - 3r)u_i^n\]
\[+ (248 + 4r)u_{i+1}^n + (-2 - r)u_{i+2}^n\]  \tag{8}
\[2ru_{i-2}^n + (124 - 7r)u_{i-1}^n + (474 + 9r)u_i^n\]
\[+(124 - 5r)u_{i+1}^n + (-2 + r)u_{i+2}^n = (124 + r)u_i^n\]
\[+(474 - 3r)u_i^n + (124 + 3r)u_{i+1}^n + (-2 - r)u_{i+2}^n\]  \tag{9}
\[(-2 + r)u_{i+1}^n + (124 - 3r)u_{i+1}^n + (474 + 3r)u_i^n\]
\[+ (124 - r)u_{i+1}^n = (-2 - r)u_i^n + (124 + 5r)u_i^n\]
\[+(474 - 9r)u_i^n + (124 + 7r)u_{i+1}^n - 2ru_{i+2}^n\]  \tag{10}
\[(-2 + r)u_{i+1}^n + (248 - 4r)u_{i+1}^n + (474 + 3r)u_i^n\]
\[(-2 - r)u_{i-1}^n + (248 + 4r)u_{i+1}^n + (474 - 9r)u_i^n + 8ru_{i+1}^n - 2ru_{i+2}^n\]  \tag{11}
\[A_{14} = \begin{pmatrix}
474 + 9r & 248 - 4r & -2 + r \\
124 - 7r & 474 + 9r & 124 - 5r & -2 + r \\
-2 + r & 124 - 3r & 474 + 3r & 124 - r \\
-2 + r & 248 - 4r & 474 + 3r & 
\end{pmatrix}
\]
\[B_1 = \begin{pmatrix}
B_{11} & O \\
O & B_{14}
\end{pmatrix}
\]
\[B_{12} = \begin{pmatrix}
474 - 9r & 248 + 4r & -2 - r \\
124 + 7r & 474 - 9r & 124 + 5r & -2 - r \\
-2 - r & 124 + 3r & 474 - 3r & 124 + r \\
-2 - r & 248 + 4r & 474 - 3r & 
\end{pmatrix}
\]
\[B_{14} = \begin{pmatrix}
474 - 3r & 248 + 4r & -2 - r \\
124 + r & 474 - 3r & 124 + 3r & -2 - r \\
-2 - r & 124 + 5r & 474 - 9r & 124 + 7r \\
-2 - r & 248 + 4r & 474 - 9r & 
\end{pmatrix}
\]

Based on (5)-(12), we construct three basic explicit computing point groups:

"\(\nu 1\)" group: eight grid points are involved, and (5)-(12) are used respectively. From (13) we have
\[U_i^{n+1} = A_i^{-1}(B_1U_i^n + F_i^n)\]  \tag{14}

Then the numerical solution at the eight grid nodes can be obtained independently.

"\(\nu 2\)" group: four inner points are involved. Let \(\overline{U}_i^{n+1} = (u_i^{n+1}, u_{i+1}^{n+1}, \ldots, u_{i+3}^{n+1})^T\), then it follows
\[A_2\overline{U}_i^{n+1} = B_2\overline{T}_i^n + F_i^n\]  \tag{15}

Here \(\overline{T}_i^n = (-2ru_{i-2}^n + 8ru_{i-1}^n, -2ru_{i+1}^n, -2ru_{i+5}^n + 8ru_{i+4}^n)^T\).

\[A_2 = \begin{pmatrix}
474 + 3r & 248 - 4r & -2 + r & 1 \\
124 - r & 474 + 3r & 124 - 3r & -2 + r \\
-2 + r & 124 - 5r & 474 + 9r & 124 - 7r \\
-2 + r & 248 - 4r & 474 + 9r & 
\end{pmatrix}
\]
\[B_2 = \begin{pmatrix}
474 - 9r & 248 + 4r & -2 - r \\
124 + 7r & 474 - 9r & 124 + 5r & -2 - r \\
-2 - r & 124 + 3r & 474 - 3r & 124 + r \\
-2 - r & 248 + 4r & 474 - 3r & 
\end{pmatrix}
\]

And we have
\[\overline{U}_i^{n+1} = A_2^{-1}(B_2\overline{U}_i^n + \overline{F}_i^n)\]  \tag{16}

"\(\nu 3\)" group: four inner points are involved. Let \(\tilde{U}_i^{n+1} = (u_i^{n+1}, u_{i+1}^{n+1}, \ldots, u_{i+3}^{n+1})^T\), then it follows
\[A_3\tilde{U}_i^{n+1} = B_3\tilde{T}_i^n + \tilde{F}_i^n\]

Here \(\tilde{T}_i^n = (-2ru_{i-2}^n + 8ru_{i-1}^n, -2ru_{i+1}^n, -2ru_{i+5}^n + 8ru_{i+4}^n)^T\).
Let $U^n = (u^n_1, u^n_2, \ldots, u^n_m)^T$. Considering under periodic boundary conditions it follows $u^n_i = u^n_{i+m}$, we can denote the alternating group explicit method I (AGEI) as follows:

$$
\begin{cases}
AU^{n+1} = BU^n \\
\tilde{A}U^{n+2} = \tilde{B}U^{n+1}
\end{cases}
$$

(17)

here $A$, $B$, $\tilde{A}$, $\tilde{B}$ are all $m \times m$ matrices.

$A = \text{diag}(A_1, A_1, \ldots, A_1, A_1)$, $\tilde{B} = \text{diag}(\bar{B}_1, \bar{B}_1, \ldots, \bar{B}_1, \bar{B}_1)$

$$
\hat{A} = \begin{pmatrix}
A_3 & A_1 & -B_4 \\
A_1 & \cdots & A_1 \\
-B_4^T & A_1 & A_2
\end{pmatrix}
$$

$$
\hat{B} = \begin{pmatrix}
B_2 & B_1 \\
B_1 & \bar{B}_1 \\
B_4 & \bar{B}_1 & B_3
\end{pmatrix}
$$

$$
\tilde{B}_1 = \begin{pmatrix}
\bar{B}_{11} & \bar{B}_{12} \\
\bar{B}_{13} & \bar{B}_{14}
\end{pmatrix}
$$

$B_1 = \begin{pmatrix}
0 & 0 & -2r & 8r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$

$$
\tilde{B}_{11} = \begin{pmatrix}
474 - 3r & 248 + 4r & -2 - r \\
124 + 7r & 474 - 3r & 124 + 3r & -2 - r \\
-2 - r & 124 + 5r & 474 - 9r & 124 + 7r \\
-2 - r & 248 + 4r & 474 - 9r
\end{pmatrix}
$$

$$
\tilde{B}_{12} = \begin{pmatrix}
0 & 0 & -2r & 8r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\tilde{B}_{13} = \begin{pmatrix}
0 & 0 & -2r & 8r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\tilde{B}_{14} = \begin{pmatrix}
474 - 9r & 248 + 4r & -2 - r \\
124 + 7r & 474 - 9r & 124 + 5r & -2 - r \\
-2 - r & 124 + 3r & 474 - 3r & 124 + r \\
-2 - r & 248 + 4r & 474 - 3r
\end{pmatrix}
$$

Case 2: Let $m - 1 = 8s + 4$, here $s$ is an integer. First at the $(n+1)$-th time level, we divide all of the $m - 1$ inner grid points into $s$ "v1" groups, $"v2"$ group are used in the right four grid points $(m-4, n+2), (m-3, n+2), (m-2, n+2), (m-1, n+2)$. By alternating use of the asymmetry schemes (5)-(12), the computing in the whole domain can be divided into many sub-domains, and grouping explicit computation can be obtained obviously. So the method has the obvious property of parallelism.

Let $U^n = (u^n_1, u^n_2, \ldots, u^n_m)^T$. Considering under periodic boundary conditions it follows $u^n_i = u^n_{i+m}$, we can denote the alternating group explicit method II (AGEII) as follows:

$$
\begin{cases}
\tilde{A}U^{n+1} = \tilde{B}U^n \\
\tilde{A}U^{n+2} = \tilde{B}U^{n+1}
\end{cases}
$$

(18)

here $\tilde{A}$, $\tilde{B}$, $\tilde{A}$, $\tilde{B}$ are all $m \times m$ matrices.

$\tilde{A} = \begin{pmatrix}
A_1 & \cdots & A_1 \\
B_1 & \cdots & \bar{B}_1 \\
\bar{B}_1 & \cdots & B_3
\end{pmatrix}$

$\tilde{B} = \begin{pmatrix}
A_1 & \cdots & A_1 \\
\bar{B}_1 & \cdots & A_2
\end{pmatrix}$
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$$\tilde{B} = \begin{pmatrix} B_3 & \tilde{B}_1 \\ \vdots & \vdots \\ \tilde{B}_1 & B_1 \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} A_3 & B_4 \\ \vdots & \vdots \\ B_1 & A_1 \end{pmatrix},$$

$$\tilde{\tilde{B}} = \begin{pmatrix} \tilde{B}_1 & \tilde{B}_1 \\ \vdots & \vdots \\ \tilde{B}_1 & B_2 \end{pmatrix},$$

$$\tilde{B}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2r & 8r \\ 0 & 0 & 0 & 0 & 0 & 0 & -2r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We point out that computation in each group can also be finished independently. So the parallelism of the AGEII method is obvious.

3 Stability and Convergence Analysis

In order to verify the stability of (17), we present the following lemma [8]:

Lemma 1 If $\text{M}=(m_{ij})$ is a $n \times n$ diagonal dominant L-matrix, while $\text{N}(n_{ij})$ is a $n \times n$ nonnegative definite matrix, then it follows:

$$\min \left( \sum_i n_{ij}/\sum_j m_{ij} \right) \leq \rho(M^{-1}N) \leq \|M^{-1}N\|_{\infty} \leq \max \left( \sum_i n_{ij}/\sum_j m_{ij} \right) \quad (19)$$

Theorem 1 The AGEI method defined by (17) is unconditionally stable.

Proof: From (17) we have $U^{n+2} = GU^n$, here $G = \tilde{A}^{-1}B\tilde{B}A^{-1}B$ is the growth matrix. From the construction of the matrices above we can see $A$ and $\tilde{A}$ are both strictly diagonally dominant L-matrices, while $B$ and $\tilde{B}$ are both nonnegative definite real matrices. Then from lemma 1 we have:

$$\rho(\tilde{A}^{-1}\tilde{B}) \leq 1, \leq \rho(A^{-1}B) \leq 1$$

Then we have $\rho(G) = \rho(\tilde{A}^{-1}\tilde{B}A^{-1}B) \leq \rho(\tilde{A}^{-1}\tilde{B})\rho(A^{-1}B) \leq 1$, which shows the AGE method (17) is of unconditional stability.

Considering $\left( \frac{\partial^k}{\partial x^k} u \right) = (-1)^k \left( \frac{\partial^k}{\partial x^k} u \right)$, applying Taylor’s formula to (5)-(12) we can easily obtain that the truncation error is $O(h^2 + h^4 + \tau h^2 + \tau h^4 + \tau^2 + h^6)$ respectively. Furthermore alternating use of (5)-(12) can lead to counteraction of the truncation error for the items containing $h^2, h^4, \tau, \tau h, \tau h^2, \tau h^3$. Then it follows the truncation error of (17) is $O(\tau^2 + h^6)$, which shows (17) is compatible with (1).

According to Lax theorem, (17) is convergent under the fact of unconditional stability. So we have:

**Theorem 2** The AGEI method defined by (17) is convergent.

Similarly we have:

**Theorem 3** The AGEII method defined by (18) is also unconditionally stable and convergent.

4 Numerical Experiments

We consider the following example:

$$\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, & 0 \leq x \leq 2\pi, \ 0 \leq t \leq T \\
u(x,0) = sin x, & \{u(0,t) = u(2\pi, t) = 0.\}
\end{cases} \quad (20)$$

The exact solution for the problem is $u(x,t) = e^{-t}sin x$. Let $||E_1||_{\infty}$ denote maximum absolute error, while $||E_2||_{\infty}$ denote maximum relevant error. $||E_1||_{\infty} = |u^n - u(x_i, t_n)|, ||E_2||_{\infty} = 100 \times |u^n - u(x_i, t_n) - u(x_i, t_n)|$. In order to verify the presented AGE method, we present the numerical results of comparisons with implicit Crank-Nicolson scheme (C-N) in the following tables:

Table 1: Numerical results at $m = 16, \ t = 100\tau$

| $\tau$ | $||E_1||_{\infty}$ | $||E_1||_{\infty}(C - N)$ | $||E_2||_{\infty}$ | $||E_2||_{\infty}(C - N)$ |
|---|---|---|---|---|
| $10^{-3}$ | 1.570 $\times 10^{-4}$ | 1.214 $\times 10^{-4}$ | 4.106 $\times 10^{-2}$ | 3.717 $\times 10^{-2}$ |
| $10^{-4}$ | 2.086 $\times 10^{-5}$ | 1.673 $\times 10^{-5}$ | 5.452 $\times 10^{-3}$ | 4.561 $\times 10^{-3}$ |

Table 2: Numerical results at $m = 24, \ t = 100\tau$

| $\tau$ | $||E_1||_{\infty}$ | $||E_1||_{\infty}(C - N)$ | $||E_2||_{\infty}$ | $||E_2||_{\infty}(C - N)$ |
|---|---|---|---|---|
| $10^{-3}$ | 5.839 $\times 10^{-5}$ | 3.479 $\times 10^{-5}$ | 1.279 $\times 10^{-2}$ | 0.931 $\times 10^{-2}$ |
| $10^{-4}$ | 1.295 $\times 10^{-5}$ | 0.976 $\times 10^{-5}$ | 5.004 $\times 10^{-3}$ | 3.547 $\times 10^{-3}$ |
Table 3: Numerical results at $m = 40$, $t = 100\tau$

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<tr>
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<th>$\tau = 10^{-5}$</th>
<th>$m = 24$, $\tau = 10^{-4}$</th>
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<td>E_1</td>
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<td>E_2</td>
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</table>

From Table 1,2,3 we can see that the present method has nearly the same accurate as the implicit C-N scheme. Furthermore, we notice the method is suitable for parallel computing.

5 Conclusions

In this paper, we present an unconditionally stable alternating group method with intrinsic parallelism for fourth order parabolic equations by use of saul’yev asymmetry schemes, and the results of Table 1-3 show that the numerical solution by the presented method is of high accuracy. The construction of the AGE method can also be applied to other problems.

References:


