Simplified SNR Degradation under the Combined Effect of Imperfections Parameters in OFDM System

Invited Paper
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Abstract: In this paper, we provide a simple closed-form expression of the effective signal-to-noise ratio (SNR) for orthogonal frequency division multiplexing (OFDM) performance under the combined effect of synchronization errors (i.e. phase noise, carrier frequency offset, Doppler spread, sampling frequency offset, and timing jitter), transmitter I/Q imbalance, receiver I/Q imbalance, and amplifier nonlinearity of the high power amplifier (HPA). This formula describes explicitly the various parameters with the system performance. SNR degradation factor is plotted versus SNR for different combinations of these imperfections. This degradation measure reflects the proper parameters selection to specify OFDM system.

Key-Words: - OFDM, synchronization errors, amplitude & phase mismatches, amplifier nonlinearity.

1 Introduction
Orthogonal frequency division multiplexing (OFDM) is quite an attractive transmission scheme for high speed communications in multipath fading. However, OFDM suffers from some disadvantages. There have been many papers studied each imperfection of OFDM separately. These studies such as [1], [2], [3] for phase noise, [4], [5] for carrier frequency offset (CFO), [6], [7] for Doppler spread, [8] for sampling frequency offset (SFO), [9] for timing jitter, [10], [11] for amplifier nonlinearity, [12] for transmitter amplitude and phase mismatches (Tx I/Q imbalance), and [13] for receiver amplitude and phase mismatches (Rx I/Q imbalance). However, few papers deal with the combined effect of these problems such as [14] for joint effect of phase noise and amplifier nonlinearity and [15] for the effect of both Tx and Rx I/Q imbalances. We derive in this paper a simplified closed-form expression of the effective SNR of all the previous problems under some parameters constrains. This paper is the extension of the previous work which dealt with the combined effect of phase noise, CFO, Doppler spread, and amplifier nonlinearity [16]. The paper is organized as follows. In section 2, the system model is presented. In sections 3 and 4, we provide the performance analysis with exact calculations and its approximations respectively. imperfections is. Finally, the paper is concluded in section 5.

2 System Model
Due to the presence of the phase noise and amplifier nonlinearity only, the transmitted signal will be [2], [16]

\[ s(t) = \frac{1}{\sqrt{N+G}} \sum_{k=-N/2}^{N-1} (\alpha.d_k + D_k)e^{j\phi(t)} e^{j2\pi k t/T} p(t) \]

where \( \alpha \) is the complex gain factor due to the amplifier nonlinearity. \( d_k \) is the data symbol which is assumed to be zero-mean random variable with variance \( E_s = E(d_k^2) \) where \( E_s \) is the symbol transmitted energy and \( E(.) \) denotes the expectation of the argument, and \( N \) is the number of the subcarriers which is equal to the number of samples in the fast Fourier transform (FFT) duration. \( G \) is the number of samples in the guard interval. \( D_k \) is the nonlinear distortion (NLD) due to HPA with variance \( \sigma_D^2 \). \( \phi(t) \) is the phase noise of the LO.
which can be described by Wiener process with zero mean and variance, $\sigma^2_{\text{wiener}}$ given by [2]

$$\sigma^2_{\text{wiener}} = 2\pi \beta |p|$$  \hspace{1cm} (2)

where the parameter $\beta$ represents the 3-dB linewidth of Lorentzian power spectral density of the LO. This model is applied when the LO is frequency-locked (free-running oscillator).

$T = T_s - T_g$ is the duration of the useful OFDM symbol (FFT duration) where $T_s$ is the transmitter OFDM symbol and $T_g$ is the duration of the guard interval, respectively. $p(t)$ is the pulse-shaping function, where for rectangular pulse $p(t) = 1$ in $[-T_g, T)$. We restricted this analysis on rectangular pulse. The detected values at the input of a decision element is given by

$$Z_k = \frac{T}{T_s}[\alpha I_0 a_k + I_0 D_k + \sum_{r=\frac{M}{2},r\neq k}^{N-1} I_{k-r}(\alpha a_r + D_r)] + N_k$$  \hspace{1cm} (3)

where $I_k$ is given by

$$I_{k-r} = \frac{1}{T_s} \int_0^T p(t) e^{j2\pi(k-r)\frac{t}{T}} dt$$  \hspace{1cm} (4)

In the presence of frequency offset ($\Delta f = f_o + f_d$, where $f_o$ is the CFO and $f_d$ is the maximum Doppler frequency), timing jitter, $\tau(t)$ and SFO ($\varepsilon = T_{\text{sample}} - T_{\text{sample}}')$, $I_{k-r}$ is modified to be [5], [6], [8], and [9]

$$I_{k-r} = \frac{1}{T_s} \int_0^T p(t)e^{j2\pi((\Delta f + \tau)\frac{t}{T})} e^{j2\pi\varepsilon \frac{r}{T}} dt$$  \hspace{1cm} (5)

where $T_{\text{sample}} = T = T_s \frac{N}{G} \frac{T_s}{N+G}$ is the sampling period and $h(t)$ is complex-valued Gaussian random process channel impulse response with zero mean and unit variance.

In the presence of both Tx and Rx I/Q imbalances, $Z_k$ will be modified to take the following form [12], [13], and [15]

$$Z_k = \mu Z_k + \nu Z_k^*$$  \hspace{1cm} (6)

where

$$\mu = \mu_r \mu_i + \nu \nu^*_r$$

$$\nu = \mu_r \nu_i + \nu_i \mu^*_r$$

and

$$\mu_r = \cos\left(\frac{\theta}{2}\right) + j\gamma^r \sin\left(\frac{\theta}{2}\right), \quad i = t(Tx) \text{ or } r(Rx)$$  \hspace{1cm} (8)

$$\nu = \gamma^r \cos\left(\frac{\theta}{2}\right) - j \sin\left(\frac{\theta}{2}\right)$$

where $\gamma^r$ is the amplitude mismatch, $\theta$ is the phase mismatch, and * stands for the conjugate.

### 3 Performance analysis

In order to derive the effective SNR, it is convenient to separate the $I_0$ in its mean value and in its varying part [5]. So, (4) will be

$$Z_k = [\mu a E(h_i) h_k + \mu a a_k (I_0 - E(h_i))] + \nu a a_k^* H_k^*$$

$$+ \nu a a_k H_k^*$$

$$+ \mu a D_k H_k^*$$

$$+ \nu a D_k H_k^*$$

$$+ \nu \sum_{r=0,r\neq k}^{N-1} I_{k-r}(a a_r + D_r) H_k^*$$

$$+ \nu \sum_{r=0,r\neq k}^{N-1} I_{k-r}(a a_r + D_r) H_k^*$$

The last equation contains nine terms. The first term is the desired term, the second term is the self-interference (SI) term, the forth and fifth terms are the NLI terms, and the third, sixth and the seventh terms are the intercarrier interference (ICI) terms. Finally, the eight and ninth terms are the additive white Gaussian noise (AWGN) terms. So, the effective SNR is given by (see the Appendix)

$$
\text{SNR}_{\text{eff}} = \frac{\text{var} \| h \|^2 D_{\text{eff}} 1 \int \frac{\beta_f^2}{T} E^2(|h|) d\beta_f}{1 + D_{\text{eff}} \sigma_N^2 + \text{var} \| h \|^2 D_{\text{eff}} 1 \int E^2(|h|) d\beta_f}.
$$  \hspace{1cm} (9)

The last equation can be further approximated to take the value

$$
\frac{1}{T} \int_{-\gamma/2}^{\gamma/2} E^2(|h|) d\beta_f \approx e^{-\alpha^2(1 - \frac{\pi^2}{3} \sigma_i^2 - \frac{\pi^2}{3} (f_d + f_o)^2 T^2 - \frac{\pi^2}{48} \epsilon^2)}
$$

$$+ \frac{\pi^2}{9} (f_d + f_o)^2 T \sigma_i^2 + \frac{\pi^4}{60} \epsilon^4 \sigma_i^2.
$$  \hspace{1cm} (10)

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$$+ \frac{\pi^2}{9} (f_d + f_o)^2 T \sigma_i^2 + \frac{\pi^4}{60} \epsilon^4 \sigma_i^2.
$$  \hspace{1cm} (11)
\[
\frac{1}{T} \int_{-T/2}^{T/2} E^2(|I_k|) \, dk \approx e^{-\alpha T} \left( 1 - \frac{\pi^2}{3} \sigma_{\epsilon}^2 - \frac{\pi^2}{3} (f_0 + f_0) T^2 - \frac{\pi^2}{48} \epsilon^2 \right) 
\]

(12)

where SNR is the signal-to-noise ratio which is equal to \( E_s / \sigma^2 \), \( \sigma_{\epsilon}^2 \) is the NLD variance, \( \sigma_D^2 \) normalized to AWGN variance, \( \sigma_\epsilon^2 \) is the timing jitter variance, \( k \) is the subcarrier index, \( D_x = \frac{T}{T_s} \) is the degradation factor due to the guard interval, and

\[
|h|^2 \equiv \left| \frac{\mu^2}{|\mu^2| + \rho} \right|^2. 
\]

(13)

In the previous analysis, we used Gaussian approximation (GA) method as was done in [2] to get \( SNR_{eff} \). Although GA is not accurate, it is acceptable for small values of frequency offset which are the practical values in OFDM systems [4]. Also, although the ICI due to phase noise is nongaussian and tends to Gaussian at high values of normalized linewidth (\( \beta T \approx 1 \)) [3], GA can be used and gets acceptable results for moderate SNR even in small values of linewidth.

The normalized NLD variance, \( \sigma_{\epsilon}^2 \), for the soft envelope limiter (SEL) power amplifier is given by [10]

\[
\sigma_{\epsilon}^2 = SNR (1 - e^{-\alpha IBO}) - \alpha^2, 
\]

(14)

\[
\alpha = 1 - e^{-\alpha IBO} + \frac{\sqrt{\pi} (IBO)}{2} \text{erfc}(\sqrt{IBO}) 
\]

(15)

where \( IBO \) is the input backoff, which represents the ratio between the input saturation power and the input mean power. It is clear from (14) and (15) that as \( IBO \) decreases (i.e. the effect of amplifier nonlinearity increases) \( \sigma_{\epsilon}^2 \) will increase and \( \alpha \) will decrease.

In the presence of one imperfection at the absence of the others, we obtain \( SNR_{eff} \) exactly as was obtained in [2], [5], [6], [8], [9], [10], and [13].

From (10)-(15), it is clear that the relationship between the effective SNR and the parameters describing the effects of phase noise, CFO, Doppler spread, SFO, timing jitter, Tx and Rx I/Q imbalances, and amplifier nonlinearity of HPA is obtained explicitly.

4 Exact calculations and its approximations

As shown in the Appendix, using (28)-(37) and substituting into (27), it is clear that we have exact closed-form expressions for desired, SI, NLI, and ICI powers, where

\[
E^2(|I_k|) = e^{-\frac{\pi^2}{3} \frac{\pi^2}{3} \epsilon^2} \sinh^2(\pi \beta T/2) \sin^2(\psi) \cosh^2(\pi \beta T/2) 
\]

(16)

\[
E(|I_k|^2) = e^{-\frac{\pi^2}{3} \frac{\pi^2}{3} \epsilon^2} \left[ \sinh^2(\pi \beta T/2) \sin^2(\psi) \cosh^2(\pi \beta T/2) \right. 
\]

(17)

where \( \psi = \pi(\Delta f T + \epsilon k + \epsilon \Delta f T) \) and \( \sin(\alpha) \equiv \sin(\alpha \pi)/\pi \).

From the previous two equations, the desired and the interference powers can be approximated with simply closed-form expressions under certain conditions as follows:

\[
E^2(|I_k|) \equiv e^{-\frac{\pi^2}{3} \frac{\pi^2}{3} \epsilon^2} e^{-\psi} \sin^2(\psi \pi/\pi), \beta T << 1, (\beta T)/\psi << 1 
\]

(18)

(19)

Averaging (18) and (19) along the subcarrier index, \( k \), and taking the following two approximations [9], [6]

\[
e^{-\frac{\pi^2}{3} \frac{\pi^2}{3} \epsilon^2} \approx 1 - 4\pi^2 \frac{k^2}{T^2} \sigma_{\epsilon}^2, 
\]

(20)

\[
\sin^2(\Delta f T + \epsilon k + \epsilon \Delta f T) \approx 1 - \frac{\pi^2}{3} \left( \Delta f T + \epsilon k + \epsilon \Delta f T \right)^2, 
\]

(21)

where (20) and (21) are very close to the exact values for \( \sigma_{\epsilon}^2 \leq 0.125 \) [9] and \( \Delta f T \leq 0.2 \) [6] respectively, the simple expressions of average powers can be obtained. Finally, \( SNR_{eff} \) will be obtained as given in (11) and (12). SNR degradation against a lot number of combinations of parameters values of imperfections can be plotted, where the SNR degradation, or degradation factor (DF), is given by

\[
DF_{db} = 10 \log \left( \frac{SNR}{SNR_{eff}} \right) 
\]

(22)
where $DF_{dB}$ is the DF in dB. As an example, Fig. 1 shows the DF versus different values of the combined parameters $(\beta T, f_0T, f_pT, \sigma^2, \gamma, \theta)$ for $IBO = 4 dB$ and $IBO = 10 dB$ respectively.

The parameter $\varepsilon$ is ignored because in practically sampling offsets of 1-10 ppm, the effect of it on $SNR_{eff}$ is very small. Also, in this figure, we ignore the effect of guard interval for simplicity (i.e. $D_g = 1$).

We see that the DF for the values shown in Fig. 1 at SNR=20 dB for IBO=10 dB will be 6.59 dB, 9.11 dB, and 9.38 dB respectively, taking into consideration that $IBO = 10 dB$ corresponds to linear power amplifier. The degradation in SNR for $IBO=4 \text{ dB}$ is raised to 7.53 dB, 9.68 dB, and 9.92 dB respectively. As a result, the effect of nonlinearity of HPA for the above values leads to small impact in SNR degradation (0.94 dB, 0.57 dB, and 0.54 dB respectively). On the other hand, as an example, in the presence of phase noise effect only $(\beta T = 0.0001, \text{not shown in the figure})$ the DF at SNR = 20 dB will be 0.04 dB and 3.16 dB for IBO=10 dB and IBO=4 dB respectively (i.e. the effect of HPA nonlinearity is increased to 3.12 dB). So, specifying the parameters values is important in designing OFDM system.

![Fig. 1. DF versus SNR for different values of the combined parameters $(\beta T, f_0T, f_pT, \sigma^2, \gamma, \theta)$ for $IBO=4 \text{ dB}$ and $10 \text{ dB}$.](image)

**APPENDIX**

### 1 Computation of the effective SNR

In order to compute $SNR_{eff}$, we must compute the desired, SI, NLI, ICI, and AWGN powers, where from (10)

$P_{des} = E_{\text{I}} |\alpha|^2 |\mu|^2 E^2 (|I_0|^2),$ 

$P_{SI} = E_{\text{I}} |\alpha|^2 |\mu|^2 (E(|I_0|^2) - E^2 (|I_0|^2)),$ 

$P_{NLI} = (|\mu|^2 + |\varepsilon|^2) \sigma^2_{\text{NLI}} E(|I_0|^2),$ 

$P_{ICI} = E_{\text{I}} |\alpha|^2 |\mu|^2 E(|I_0|^2) + (|\mu|^2 + |\varepsilon|^2) (E_{\text{I}} |\alpha|^2 + \sigma^2_{\text{ICI}}) E(\sum_{k=0}^{N-1} |I_{k+r}|^2)$

and

$P_{\text{AWGN}} = (|\mu|^2 + |\varepsilon|^2) \sigma^2$  (23)
where, from (5), $E(\|I_0\|^2)$ is given by

$$E(\|I_0\|^2) = E\left(\frac{1}{T} \int_{-T/2}^{T/2} e^{j2\pi k f_T} e^{j2\pi \tau(t)} dt \right)$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} E(e^{j2\pi \tau(t)}) e^{j2\pi \tau(t)} dt \tag{24}$$

where $E(e^{j\theta(t)})$ is given by [2] as follow

$$E(e^{j\theta(t)}) = e^{-\sigma^2 |\theta|} \tag{25}$$

and $\tau(t)$ is a (WSS) Gaussian random process with expectation given by [9] as follow

$$E(e^{j2\pi \tau(t)/T}) = e^{-\frac{k^2 T^2}{\sigma^2}}. \tag{26}$$

From (25) and (26) into (24), and after some algebra, we obtain (16).

$$E\left(\sum_{r=0}^{N-1} |I_{k+r}|^2\right) = E\left(\sum_{r=0}^{N-1} |I_r|^2\right) = E(\|I_0\|^2) \tag{27}$$

where $E(\|I_0\|^2)$ is given by [17,10.27]

$$E(\|I_0\|^2) = \frac{1}{T} \int_{-T/2}^{T/2} \left|\xi(t)\right|^2 e^{j2\pi \xi(t)} R_{\xi}(T) R_{\eta}(T) dx \tag{28}$$

where $R_{\xi}(x)$ is the autocorrelation function of the phase noise, which is given by [2]

$$R_{\xi}(x) = e^{-\sigma^2 |\theta|} \tag{29}$$

where $\xi(t) \equiv e^{j\theta(t)}$. $R_{\eta}(x)$ is the autocorrelation function of the frequency offset which is given by

$$R_{\eta}(T) = \cos(2\pi \Delta f T) \tag{30}$$

where, we use the two-path model for the Doppler spectrum model which represents the worst extreme condition [6]. Two-path model corresponds to a system with a fixed frequency offset. So, in the presence of both CFO and Doppler spread, we can combine them with frequency offset $\Delta f = f_o + f_d$.

$R_{\xi}(x)$ is the autocorrelation function of the uncorrelated timing jitter, which is given by [9]

$$R_{\xi}(x) = e^{-4\sigma^2 |\eta|^2} \tag{31}$$

Using the formula [18,4-4-16, 4-4-17]

$$\sum_{r=-\infty}^{\infty} e^{-j2\pi x r} = \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{32}$$

thus, from (29)-(32) into (28), and after some algebra, we obtain

$$E\left(\sum_{r=0}^{N-1} |I_{k+r}|^2\right) = 1 - E(\|I_0\|^2) \tag{33}$$

Substituting from (29)-(31) into (28), putting $r = 0$, and after some algebra, we get $E(\|I_0\|^2)$ as given by (17). In the previous calculations, we normalized the channel, i.e

$$E(\|h(t)\|^2) = 1 \tag{34}$$

From (16), and (17), it is clear that the desired and the total interference (TI=SI+NLI+ICI) powers depend on the subcarrier index, $k$. So, we must average them, where the average desired and TI powers are obtained by

$$\bar{P}_{\text{des}} = \frac{1}{T} \int_{-T/2}^{T/2} P_{\text{des}} dk,$$

$$\bar{P}_{\text{TI}} = \frac{1}{T} \int_{-T/2}^{T/2} P_{\text{TI}} dk. \tag{35}$$

So $\text{SNR}_{\text{eff}}$ will be

$$\text{SNR}_{\text{eff}} = \frac{\bar{P}_{\text{des}}}{\bar{P}_{\text{TI}} + P_{\text{AWGN}}} \tag{36}$$

From (20), (21), and (35) into (36), we get (11).

References:


