Finite Element Analysis of Quasistatic Frictional Contact Problems with an Incremental-Iterative Algorithm

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Abstract: In this paper, we consider numerical approximations of 3D quasistatic contact problem with dry friction, using finite contact elements. Lagrange incremental multipliers method and penalty functions are used to enforce finite element surface contact constrains for incremental-iterative formulation of the quasistatic problem.

Key–Words: Quasistatic 3D contact problem, Coulomb dry friction, finite element, Lagrangian perturbed, variational inequations

1 Introduction

The importance of contact processes in structural mechanic and mechanical systems is given by a big effort that has been made for modelling and numerical simulations and by the extensive literature concerning this topic. In this paper we deal with the numerical analysis of a quasistatic contact problem in linear elasticity with dry friction. The problem intends to model the physical situation of two elastic bodies (or an body and his foundation) that come in contact with friction obeying the normal compliance. We consider the discrete variational formulation of incremental problem using a perturbed Lagrangian functional. The algorithm use the backward finite difference approximation, finite element method and Newton method for the linear iterations. The main purpose of the paper is to use a new contact finite elements in three-dimensional case, generalizing the two dimensional case considered by Ju and Taylor [7] and by Wriggers and Simo [16].

2 Classical Formulation of the Quasi-Static Problem

The quasistatic problem is to find the field of displacements $\mathbf{u} = (u_1, \ldots, u_d)$, defined on $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, for a time interval $[0, T]$, which satisfy the following equations and conditions:

- the equilibrium equation
  \begin{equation}
  \sigma_{ij,j}(\mathbf{u}) + f_i = 0 \quad \text{in} \quad \Omega \times [0, T] \tag{1}
  \end{equation}

- the constitutive equation
  \begin{equation}
  \sigma_{ij}(\mathbf{u}) = a_{ikh} \varepsilon_{kh}(\mathbf{u}) \quad \text{in} \quad \Omega \tag{2}
  \end{equation}

where $a_{ikh} = a_{jik} = a_{khi}$ and $a_{ikh} \varepsilon_{kh} \geq c|\xi|$, $\xi = (\xi_{ij})$ and

\[ \varepsilon_{kh}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k} \right), i, j, k, h = 1, \ldots, d; \]

- the boundary conditions
  \begin{equation}
  \mathbf{u} = 0 \quad \text{on} \quad \Gamma_0 \times [0, T] \tag{3}
  \end{equation}

\[ \sigma_{ij}(\mathbf{u}) n_j = h_i \quad \text{on} \quad \Gamma_1 \times [0, T]; \tag{4} \]
- the initial conditions
\[ u(x, 0) = u_0, \quad \dot{u}(x, 0) = u_1 \quad \text{in} \quad \Omega \tag{5} \]
at \( t = 0 \) with \( u_0, u_1 \) given smooth functions of \( x \);
- the normal compliance response
\[ \sigma_n(u) = c_N(u_N - g)_{mN} \quad \text{on} \quad \Gamma_C \times [0, T]; \tag{6} \]
- the friction and contact conditions:
\[ u_N \leq g, \quad \sigma_N \leq 0, \quad \sigma_N(u_N - g) = 0, \tag{7} \]
\[ |\sigma_T(u)| < c_T(u_N - g)_{mT} \Rightarrow \dot{u}_T = 0, \]
\[ |\sigma_T(u)| = c_T(u_N - g)_{mT} \Rightarrow \exists \lambda \geq 0 \quad \dot{u}_T = -\lambda \sigma_T \tag{8} \]
where \( c_N, m_N, c_T, m_T \) are material constants depending on interface properties, \( b_+ = \max(b, 0) \), \( \dot{u}_T \) is the tangential velocity of material particles on \( \Gamma_C \) and \( g, g \geq 0 \) is the initial gap between \( \Gamma_C \) and the foundation measured along the outward normal direction to \( \Gamma_C \).
The friction law (6)-(8) is a generalization of the Coulomb’s law, which is recovered if \( m_N = m_T \). In such case, \( \mu = c_T / c_N \) is the usual coefficient of friction. This law describes a dependence of the friction coefficient on normal contact pressure.

The classical formulation of the quasi-static contact problem is as follows:

**Problem 1.** Find the displacement field \( u \) which satisfy the relations (1)-(8) for \( \forall t \in [0, T] \), where \( f(f_i) \) and \( h(h_i) \) are taken time dependent.

### 3 Variational formulation of the problem

We introduce the following notations \( V = \{ v \in [H^1(\Omega)]^d \ \text{a.e. on} \ \Gamma_0 \} \), \( \| \cdot \| \) shall denote the norm introduced by ( ), the scalar product on \( V, K = \{ v \in V; v_N \leq g \ \text{a.e. on} \ \Gamma_C \} \) and \( W = W^{1,2}(0, T; V) \).

We suppose that \( f \in W^{1,2}(0, T; L^2(\Omega)^d), h \in W^{1,2}(0, T; L^2(\Gamma_1)^d), g \in H^{1/2}(\Gamma_C), C_N, C_T \in L^\infty(\Gamma), 1 \leq m_N, m_T, \) when \( d = 2, 1 \leq m_N, m_T \leq 3, \) when \( d = 3. \)
The variational formulation of the Problem 1 is the Problem 2:

**Problem 2.** Find \( u = (u_1, \ldots, u_d)(0, T; L^2(\Omega)^d) \) such that \( \forall t \in [0, T] \ \text{a.e.} \ u(t) \in K \) and
\[ a(u(t), v - \dot{u}(t)) + Vj_n(u(t), v - \dot{u}(t)) + j_l(u(t), v) \]
\[ -j_l(u(t), \dot{u}(t)) \geq L(v - \dot{u}(t)), \quad \forall \ v \in V, \tag{9} \]
with the initial conditions
\[ u(x, 0) = u_0, \quad \dot{u}(x, 0) = u_1, \tag{10} \]
where \( u_0, u_1 \in K \) are given and
\[ a(u, v) = \int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v) d\Omega \tag{11} \]
is the virtual work produced by the action of the stress \( \sigma_{ij} \) on the strains \( \varepsilon_{ij} \);
\[ j_n(u, v) = \int_{\Gamma_C} c_N(u_N - g)^{m_N} v_N ds \tag{12} \]
is the virtual work produced by the normal pressure in the displacement \( v; \)
\[ j_l(u, v) = \int_{\Gamma_C} c_T(u_N - g)^{m_T} ||v_T|| ds \tag{13} \]
is the virtual work produced by the frictional force on the velocity \( v; \)
\[ L(v) = \int_\Omega f v d\Omega + \int_{\Gamma_1} h v d\Gamma \tag{14} \]
is the virtual work produced by the external forces.

### 4 Incremental formulation

Now we derive a time discretized approximation of the quasi-static Problem 2. Let us consider a partition \((t_0, t_1, \ldots, t^n)\) of the time interval \([0, T]\) and an incremental formulation, obtained by using the backward finite difference approximation of the time derivative of \( u \).

If we set \( u^k = u(x, t^k), \Delta u^k = u^{k+1} - u^k, \Delta t^k = t^{k+1} - t^k, \quad L^k = L(u^k), \Delta L^k = L^{k+1} - L^k, \quad k = 0, 1, \ldots, n - 1, \) and if we take \( \dot{u}(t^{k+1}) = \Delta u^k / \Delta t^k \) we obtain, at each moment \( t^k \), the following quasi-variational inequalities:

**Problem 3.** Find \( \Delta u^k \in K \) such that
\[ a(\Delta u^k, v - \Delta u^k \Delta t^k) + j_n(\Delta u^k, v - \Delta u^k \Delta t^k) \]
\[ + j_l(\Delta u^k, v) - j_l(\Delta u^k, \Delta u^k \Delta t^k) \]
\[ \geq L^k(\Delta u^k, v - \Delta u^k \Delta t^k)\]
\[ - F^k(\Delta u^k, v - \Delta u^k \Delta t^k) \quad \forall \ v \in V, \]
where
\[ F^k(\Delta u^k, v - \Delta u^k \Delta t^k) = a(\Delta u^k, v - \Delta u^k \Delta t^k) - L^k(v - \Delta u^k). \]

The other form of the time-discretized approximation of the problem is as follows:

**Problem 4.** Find \( u^k \in K \) defined by \( u^0 = u_0 \), the approximate solution of a sequence of incremental Problems 3. Although every Problem 3 is a static one, it requires appropriate updating of the displacements and the loads after each increment.
5 Existence of a discrete solution

For each static Problem 4 we consider an internal approximation \((V_h, K_h, j_{nh}, j_{th})\) and we formulate the following discrete problem:

**Problem 5.** Find \(u_i^h \in K_h\) such that

\[
\begin{align*}
 a(u_i^h, v_h - u_i^h) + j_{nh}(u_i^h, v_h - u_i^h) + j_{th}(u_i^h, v_h)
 - j_{th}(u_i^h, v_h) & \geq L_i^h(v_h - u_i^h) - F_i^h(u_i^h, v_h - u_i^h), \\
 & \forall v_h \in V_h.
\end{align*}
\]

The solution \(u_i^h\) of Problem 4 is the fixed point of the mapping \(s_h : K_h \rightarrow K_n\) which associates to every \(v_i^h \in K_h\) the element \(S_h w_i^h \in K_h\) defined by:

\[
\begin{align*}
 a(S_h w_i^h, v_i^h - S_h w_i^h) + j_{nh}(S_h w_i^h, v_i^h - S_h w_i^h) + \\
 + j_{th}(S_h w_i^h, v_h) & \geq L_i^h(v_i^h - S_h w_i^h) - F_i^h(S_h w_i^h, v_h - S_h w_i^h), \\
 & \forall v_i^h \in V_i.
\end{align*}
\]

so that \(u_i^h\) is the limit of a sequence \((u_{ih})_n\) with \(u_{ih} = S_h w_{i(n-1)}^h\). This has been proved in [4], for a friction coefficient small enough to ensure that the above inequality is true.

6 Finite element method of the 3D frictional contact problems

We consider a discrete variational formulation of the incremental Problem 3, using for the contact area a three nodes contact element for the two dimensional case (see [7], [16]).

In the three dimensional case a four node contact element consisting of three "master" nodes, belong to \(\Omega^1\) (or \(\Omega\) and one "slave" node, belong to \(\Omega^2\) (respectively rigid support), is employed.

In all numerical applications we derived a perturbed Lagrangian formulation for the case of frictional stick and for the case of frictional slide. For the case of frictional stick the perturbed Lagrangian functional for bodies in contact has the following form:

\[
\begin{align*}
 & \Lambda(u, \Sigma_n, \Sigma_t, \Sigma_r) = \frac{1}{2} a(u, u) - L(u) \\
 & + \Sigma_n^T G_n + \Sigma_t^T G_t + \Sigma_r^T G_r - \frac{1}{\omega_n} \Sigma_n \Sigma_n - \frac{1}{\omega_t} \Sigma_t \Sigma_t - \frac{1}{\omega_r} \Sigma_r \Sigma_r,
\end{align*}
\]

where \(u\) is the vector of nodal displacement, \(\Sigma_n, \Sigma_t, \Sigma_r\) the vectors of normal and tangential nodal contact forces, respectively, \(G_n, G_t, G_r\) are the vectors of normal and tangential nodal gaps and \(\omega_n, \omega_t, \omega_r\) the normal and tangential penalty parameters respectively. The Newton-Raphson method was applied to the discrete variational formulations that can be derived from these perturbed Lagrangian functional.

The normal vector on defined plane by the nodes 1, 2 and 3, belong to \(\Omega^1\) (or \(\Omega\)), and respectively vectors, defined by directions of the node 1-2 and 1-3 will be:

\[
\begin{align*}
 n &= \frac{(x_2-x_1)(x_3-x_1)}{|(x_2-x_1)(x_3-x_1)|}, \\
 \tau &= \frac{x_3-x_1}{|x_3-x_1|},
\end{align*}
\]

where \(x_1 = X_1 + u_1, x_2 = X_2 + u_2, x_3 = X_3 + u_3\) signify the current positions of master nodes; \(X_1, X_2, X_3\) are reference coordinates and \(u_1, u_2, u_3\) are current nodal displacements of points 1, 2 and 3. In addition, we define the current "surfaces coordinates" as following:

\[
\begin{align*}
 a_t &= \frac{x_s - x_1}{|x_2 - x_1|} - \frac{x_s - x_1}{|x_3 - x_1|}, \\
 a_r &= \frac{x_s - x_1}{|x_2 - x_1|},
\end{align*}
\]

in which \(x_s = X_s + u_s\) denotes the current position of the slave node \(s\) belong to \(\Omega^2\) (or rigid support). The normal and tangential gaps \(g_n, g_t, g_r\) are defined as:

\[
\begin{align*}
 g_n &= (x_s - x_1)n, \\
 g_t &= (a_t - a_t^0)|x_2 - x_1|, \\
 g_r &= (a_r - a_r^0)|x_3 - x_1|,
\end{align*}
\]

where \(a_t^0\) and \(a_r^0\) are the old surface coordinates at the last time step known.

Note that the gap \(g\) depends on the slave node \(s\) as well as on the master nodes 1, 2 and 3. Thus, the variation of the gap is obtained according to

\[
g = \frac{d}{d\alpha} g(x_s + \alpha \eta_s, x_1 + \alpha \eta_1, x_2 + \alpha \eta_2, x_3 + \alpha \eta_3)
\]

where

\[
\eta(\eta_1, \eta_2, \eta_3, \eta_s) \equiv \delta u(\delta u_1, \delta u_2, \delta u_3, \delta u_s).
\]

With respect to finite element implementations, explicit matrix expressions for the Lagrangian multiplier formulation and the penalty formulation are derived as follows.

The discrete variational equation associated with (16) take the form:

\[
\delta u \Pi(u) + \Sigma_n^T \delta u G_n + \Sigma_t^T \delta u G_t + \Sigma_r^T \delta u G_r + 0 = 0
\]

\[
\delta \Sigma_n^T \left( - \frac{1}{\omega_n} \Sigma_n + G_n \right) = 0
\]

\[
\delta \Sigma_t^T \left( - \frac{1}{\omega_t} \Sigma_t + G_t \right) = 0
\]

\[
\delta \Sigma_r^T \left( - \frac{1}{\omega_r} \Sigma_r + G_r \right) = 0,
\]

where \(\Pi(u) = \frac{1}{2} a(u, u) - L(u)\) is the total potential energy of the bodies in contact,
To apply Newton’s iteration scheme, consistent energy of the contacting bodies simply read, result
\[
K_R \delta g =\frac{\partial}{\partial u_3} \eta_3^+ + \frac{\partial}{\partial u_3} \eta_3^- + \frac{\partial}{\partial u_3} \eta_3^0,
\]
where
\[
A_1 = K_B + \sum_{s=1}^{S} (k^s_n + k^s_t + k^s_r), \quad A_2 = \sum_{s=1}^{S} c^s_n,
\]
\[
A_3 = \sum_{s=1}^{S} c^s_t, \quad A_4 = \sum_{s=1}^{S} c^s_r,
\]
\[
B_2 = -\frac{1}{\omega_t}, \quad C_3 = -\frac{1}{\omega_t}, \quad D_4 = -\frac{1}{\omega_t},
\]
\[
R_1 = R_B + \sum_{s=1}^{S} (\sigma_n^s c^s_n + \sigma_t^s c^s_t + \sigma_r^s c^s_r)
\]
\[
R_2 = -\frac{1}{\omega_n} \Sigma_n + G_n, \quad R_3 = \frac{1}{\omega_t} \Sigma_t + G_t,
\]
\[
R_4 = -\frac{1}{\omega_r} \Sigma_r + G_r,
\]
\[O\] is the matrix zero, and
\[
(k^s_n)_{ji} = \frac{\partial^2 g^s_n}{\partial u_j \partial u_i}, \quad (k^s_t)_{ji} = \frac{\partial c^s_t}{\partial u_j}, \quad \frac{\partial^2 g^s_t}{\partial u_j}.
\]
Finally after the discrete formulation within the framework FEM, a standard assembly procedure can be used to add the contact contributions of each contact element to the global tangent stiffness and residual matrix and thus we obtain:
\[
K = K_B + \sum_{s=1}^{S} K^s_C, \quad R = - \left( R_B + \sum_{s=1}^{S} R^s_C \right),
\]
where
\[
K_B, \quad R_B \quad \text{are mechanical global tangent stiffness matrix and residual vector,} \quad K^s_C, \quad R^s_C \quad \text{are mechanical contact contributions of contact nod} \ s, \ U = (\Delta u, \Delta \Sigma_n, \Delta \Sigma_t, \Delta \Sigma_r)^T, \ S \quad \text{is the number of the slave nodes.} \quad \text{And for} \quad \omega_n = \omega_t = \omega_r = \omega, \quad \text{and} \quad \sigma = \omega g_n, \ \sigma_t = \omega g_t, \ \sigma_r = \omega g_r \quad \text{result}
\]
\[
K_C = \sum_{s=1}^{S} \omega (g^s_n k^s_n + g^s_t k^s_t + g^s_r k^s_r + c^s_T c^s_n + c^s_T c^s_t + c^s_T c^s_r)
\]
\[
R_C = \sum_{s=1}^{S} \omega (g^s_T c^s_n + g^s_T c^s_t + g^s_T c^s_r).
\]
For the case of frictional slide we used the relation
\[
|\Sigma_{tan}| = \mu |\Sigma_n|, \quad \text{where} \mu \quad \text{is the coefficient of friction}
and $\Sigma_{\text{tan}}$ is the result force of the $\Sigma_{t}$ and $\Sigma_{r}$, forces in the tangent plane of the contact surface.

Note with $\beta$ the angle between the sides $x_2 - x_1$ and $x_3 - x_1$; we obtain $\cos \beta = t \tau$ and

$$\lambda_{\text{tan}} = \mu \sqrt{g_l^2 + g_r^2 + 2 \varepsilon |g_t| |g_r| \cos \beta}$$

where $\varepsilon = \text{sgn} \left( g_l g_r \right)$. As a direct consequence of Coulomb's friction law, it results $\mu \omega \omega_n = \omega_r$, where $r = \sqrt{g_l^2 + g_r^2 + 2 \varepsilon |g_t| |g_r| \cos \beta}$ therefore $\lambda_t = \lambda \tan \frac{\omega_n}{r} \omega g_n = -\mu \text{sgn} \left( g_l \right) \frac{\omega_l}{r} \omega g_n = -\mu \frac{|g_l|}{r} \omega g_n$, $\lambda_r = -\mu \frac{|g_r|}{r} \omega g_n$. If we write $d_t = \frac{|g_l|}{r}$, $d_r = \frac{|g_r|}{r}$, from linearized kinematics (i.e. by neglecting non-linear terms $k_t$ and $k_r$), we obtain

$$K_C = \sum_{s=1}^{S} \left( SL_1^s + SL_2^s \right)$$

and

$$R_C = \sum_{s=1}^{S} \omega \left( \mu g_n d_t^s c_t^s + \mu g_n d_r^s c_r^s - g_n^s c_t^s \right)$$

7 The algorithm

The Newton-Raphson method was applied to the discrete variational formulation that can be derived from these perturbed Lagrangian functional.

The scheme of solving the linearized problem is following:

(i) initialisation set the iterative count $k = 0$, $u(0) = 0$;

(ii) compute the system stiffness and residual excluding contact;

(iii) compute contact stiffness and residual for each finite element $s$, $s = 1, \ldots, NC$,

a) compute the normal gap $g_n^{(s)}(k)$;

b) check for contact finite element status:

IF $g_n^{(s)}(k) > TOL$ then out of contact

ELSE in contact. Check for frictional stick or slip; compute $\sum_{t}^{(k)}$, $\sum_{r}^{(k)}$ and $\sum_{\tau}^{(k)}$

IF $\sqrt{\left( \sum_{t}^{(k)} \right)^2 + \left( \sum_{r}^{(k)} \right)^2 + \left( \sum_{\tau}^{(k)} \right)^2} \leq \mu \sum_{t}^{(k)}$

frictional stick;

ELSE frictional slip;

ENDIF

c) compute the total stiffness and residual;

(iv) solve the total system to obtain displacement increment $\Delta u^{(k)}$;

(v) check for convergence:

IF $|\Delta u^{(k)} - \Delta u^{(k+1)}| < \text{TOL1}$ then convergence and exit.

ELSE go to step (vi);

(vi) update the displacement field $u^{(k+1)} = u^{(k)} + \Delta u^{(k+1)}$;

(vii) set $k = k + 1$ and go to back to (ii).

Several examples have been taken from other works to be compared and the results obtained by us agreed to the ones (e.g. with [11]).

8 Numerical simulations

To show the performance of the numerical method described in the previous section, we have done a numerical experiment for solving a 3D frictional contact problem. We consider a three-dimensional assembling as shown in Fig. 8.1. The comparison between numerical and experimental results is presented in the Table 8.1.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$F$</th>
<th>$u_N$ calculated</th>
<th>$u_N$ measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>2000</td>
<td>0.75</td>
<td>0.79</td>
</tr>
<tr>
<td>0.1</td>
<td>2000</td>
<td>0.758</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Table 8.1

Fig. 8.1. The geometry and loading

Remark 1 The critical situations arise in transitions from sliding to adhesion because it is then that the most important changes in the solution occur. On simple remedy for these difficulties is to decrease the time step until two successive solutions are not too far apart.
References:


