Effect of Artificial Viscosity on Central Finite Volume Solution of Time-Dependent Concentration Diffusion

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Abstract

Artificial viscosity operators may be used for stabilizing the explicit solution of central type finite volume method. In present paper the effect of artificial viscosity on accuracy of numerical solution of time dependent diffusion of a concentration is investigated. This effect is investigated via reduction of the coefficients of artificial viscosity operators. A bench mark test of time dependent concentration diffusion in a channel with constant velocity and a single concentration source term with available analytical solution is used for evaluation of the numerical results.

Key words: artificial viscosity coefficient, pollutant diffusion, modeling, finite volume

1. INTRODUCTION

During recent decades, different methods of numerical modeling have been used by scientists and researchers. There are always unavoidable numerical oscillations in explicit solution of central schemes. Instead, there are methods for controlling numerical oscillations and getting stable and accurate results. One of these methods is adding artificial viscosity operators to the central type formulations [1-4]. Tuning the coefficients of such methods to the minimum required value may minimize the damping error which associates with application of these methods.

In the present work, the effect of adding artificial viscosity operator on accuracy of concentration diffusion solution is assessed by using various values of artificial viscosity coefficient for some diffusion coefficient values. The numerical solver uses central type numerical scheme finite volume scheme formulation which is developed for unstructured meshes.

For this propose a bench mark test of time dependent concentration diffusion in a channel with constant velocity and a single concentration source term with available analytical solution is used for evaluation of the numerical results. For numerical modeling of the case, shallow water equations are used to simulation of flow field and a time dependent two dimensional convection diffusion equation a concentration is solved in a coupled fashion.

2. Governing Equations

Two dimensional depth averaged hydrodynamics, including continuity and movement in horizontal plane equations, are known as shallow water equations. These equations are obtained by integrating Navier-Stokes equations from bottom to sea surface and involving bottom and surface boundary
conditions. This scheme is formed by following assumptions:

- Velocity distribution along depth of current is steady.
- Pressure distribution is hydrostatic

Continuity equation for incompressible current without considering precipitation, evaporation and bottom leakage is written as follows:

$$\frac{\partial}{\partial t}(h) + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

Where $h$ is the flow depth, $u$ and $v$ are horizontal velocities in $x$ and $y$ directions.

The depth average momentum equation in $I$ direction may be written as:

$$\frac{\partial}{\partial t}(hu_I) + \frac{\partial}{\partial x}(hu_x u_I) + h \frac{\partial}{\partial y}(gh) = -gh S_i$$

In which $S_i$ denotes the summation of bed slope and energy loss due to friction effects.

Two dimensional diffusion equation shown below is obtained:

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(huc) + \frac{\partial}{\partial y}(hvc) = \frac{\partial}{\partial x}(hD_c \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(hD_c \frac{\partial c}{\partial y})$$

Where, $c$, $u$, $v$ are the concentration and average velocities. $D_c$ is horizontal eddy diffusivity coefficient and $h$ is the sea depth. [1], [2], [3]

3. Numerical Formulation

One of the most accurate schemes for implicit solving an equation is finite volume. If this scheme is used in an unstructured triangular mesh, it will be easier to cover the solution domain and enable arbitrary and complex geometries to be replicated. Using nearly non-stretched triangular meshes also avoids interpolating from the same derivatives on the centroid nodes located on two sides of the edge in a stretched cell. In a finite volume cell centre scheme, each triangle is considered as a control volume and the state variables are located at its centroid, so that the number of unknown vectors would be the same as the number of cells or triangles. Considering the governing equations in form of an advection-diffusion type equation, $W$ represents the variables using $h$ flow depth, $u$ and $v$ the horizontal components of velocity. $G^c$ and $F^c$ are vectors of convective fluxes, while, $G^d$ and $F^d$ are vectors of diffusive fluxes of $W$ in $x$ and $y$ directions, respectively. The vector $S$ contains the sources and sinks of the governing equations covering all algebraic terms.

$$\frac{\partial W}{\partial t} + \left( \frac{\partial F^c}{\partial x} + \frac{\partial G^c}{\partial y} \right) = \left( \frac{\partial F^d}{\partial x} + \frac{\partial G^d}{\partial y} \right) + S$$

When using centroid nodes in finite volume method the equations are interrupted like follow:

$$W^{i+\Delta t} = W^i - \frac{\Delta t}{\Omega_i} \sum_{k=1}^{N_{sides}} \left[ (F^c \Delta y - \bar{G}^c \Delta x) \right] k + \Delta t S^i$$

In this equation, $W_i$ represents conserved variables at the center of control volume($\Omega_i$). $F^c$ and $\bar{G}^c$ are the mean values of convective fluxes in control volume boundaries. $S^i$ is the source term which covers volumetric evaporations and body forces. $F^d$ and $G^d$ refer to flux diffusion and are obtained by integrating around the center of control volume. The residual term includes both convection and diffusion terms:

$$R(W_i) = \sum_{k=1}^{N_{sides}} \left[ (F^c \Delta y - \bar{G}^c \Delta x) - (F^d \Delta y - \bar{G}^d \Delta x) \right] k$$

In smooth parts of the flow domain, where there is no strong gradient of flow parameters, the convective part of the residual term dominates. Since, in the explicit computation of convection dominated flow there is no mechanism to damp out the numerical oscillations; sometimes it could be
more efficient to apply numerical techniques to overcome instabilities with minimum accuracy degradation. In the present work, the artificial dissipation terms suitable for the unstructured meshes are used to stabilize the numerical solution procedure. In order to damp unwanted numerical oscillations, a fourth-order artificial dissipation term is added to the above algebraic formula:

\[ D(W_i) = S_h \varepsilon_2 \nabla^2 W_i + \varepsilon_4 \sum_{k=1}^{N_{sides}} \lambda_{ij} (\nabla^2 W_j - \nabla^2 W_i) \]

\( \lambda_{ij} \) is a scaling factor which is computed using the maximum value of the spectral radii of every edge connected to node \( i \) (\( \varepsilon_2 < 0.25 \) and \( \frac{1}{256} \leq \varepsilon_4 \leq \frac{3}{256} \)). The Laplacian operator at every node \( i \), is [4], [5]:

\[ \nabla^2 W_i = \sum_{j=1}^{N_{edges}} (W_j - W_i) \]

Here, \( 0 \leq S_h \leq 1.0 \), is a water surface waves switch which is zero for the flow regions with smooth variation of depth and may reach unity for the zones with sharp gradients of water surface elevations (ie shock waves).

4. Model Validation

Comparisons of the simulated results against analytical solution for horizontal distribution of time-dependant pollution diffusion in a river are used to show the accuracy of the developed model and its sensitivity to the diffusion coefficient values.

In the verification test case, it is supposed that the pollutant of 10g per second is continuously thrown into the middle of a channel (width 40m and depth in infinity) as shown in Fig.1. The velocity \( u \) of the channel is 1.0 m/s and the diffusion coefficient \( D_c \) in the direction of \( x \) is 1.0 m²/s. The source point is located in \( (x=0, y=0) \).

When the pollutant is released constantly and continuously along the \( z \) axis, the analytical solution of diffusion concentration at the point \( (x, y) \) after \( t \) second is shown as follows [1]:

\[ c = \frac{m}{4\pi D} \exp \left( \frac{ux}{2D} \right) \int_{-\infty}^{\infty} \exp \left( -\varphi \right) \frac{u^2 (x^2 + y^2)}{16D^2\varphi} \varphi^{-1} d\varphi \]

Where, \( m \) is the polluted loading amount per unit time and length, \( u \) is the flow velocity, \( D_c \) is the diffusion coefficient and \( \varphi \) is dimensionless integrating parameter. Diffusion distribution of the pollutant after 60 seconds is shown in Figure 2.

Figure 3 shows the distribution of the pollutant at the similar conditions is also computed using the described finite volume modeling method (using the second order artificial viscosity coefficient as \( \varepsilon_2=0.0 \) and the fourth order artificial viscosity as \( \varepsilon_4=0.025 \)). As can be seen there are negligible numerical errors in computed concentration patterns.
5. Effects of artificial viscosity coefficient

In this section the performance of the developed model to various values of the diffusion coefficient and the conflicts with the second order artificial viscosity coefficient as $\varepsilon_2$ and the fourth order artificial viscosity as $\varepsilon_4$ are investigated. Figures 4 to 6 presents the results of this sensitivity analysis.

In Fig.4, blue-dashed line and green line are almost on each other. It means when $\varepsilon_4=0$, $\varepsilon_2$ coefficient has no effect at all. Also, second and fourth order artificial viscosities ($\varepsilon_2$ and $\varepsilon_4$) only affect near source point area, where concentration gradient is maximum. Far from source point is increased, the trend of lines goes as smoother and very close to each other.

Taking a glance to Figure 4, it can be realized that the artificial viscosity coefficients have not a strong effects on the accuracy of time-dependant diffusion problems, especially when a large values of physical diffusion is involved. The computed results may have less than 1% errors in whole domain area. However for diffusion problems with small values of diffusion coefficient, the artificial viscosity may become more important and effective by decreasing peak of concentration.

In order to compare the effects of using different values of artificial viscosity on a numerical solution, in Fig.5, cross-sectional areas for different values given to second and fourth order artificial viscosity ($\varepsilon_2$ and $\varepsilon_4$) are drawn. Because of large amount of physical diffusion, the lines are overlapped.

Physical diffusion coefficient varies from one material to another. For concentration materials with physical diffusion larger than 1.0 m²/s ($D_c \geq 1.0$), artificial viscosity coefficients are almost inefficient (Fig.5).

According to Fig.6 increasing the fourth order artificial viscosity ($\varepsilon_4$) results a reduction noises and makes the results symmetric. However, the peak of the distribution curve descends. Although the surface covered by the curve is almost constant (the continuity is satisfied), the simulation distribution curve may differ from the expected results.
In the next stage, considering a certain value of physical diffusion \((D_c = 0.5 \text{ m}^2/\text{s})\), the quality of computed concentration patterns for various values of the fourth order artificial viscosity are compared with the results of analytical solution (Fig. 7).

For the cases that no artificial viscosity coefficients are applied \((\varepsilon_2 = \varepsilon_4 = 0)\), numerical errors may give rise to oscillations and a rough gradient around source point (particularly, upstream of the source point) would appear. Furthermore, distorted distribution of the concentration may appear over the modeling area and increasing the value of \(\varepsilon_2\) has no effect on concentration distribution (Fig. 8).

By increasing the value of fourth order artificial viscosity coefficient, the unwanted distributions are removed. Application of larger values of the fourth order artificial viscosity may produce smoother distribution of the concentrations (Fig. 9 to Fig. 11).

Minimum value of the fourth order artificial viscosity is \(\varepsilon_4 = 0.0125\) which produces smooth and accurate results (Fig. 10). Using excessive values for the fourth order artificial viscosity coefficient may produce smooth results but degrades the accuracy by reducing the maximum values along the channel axis (Fig. 11).
6. CONCLUSION

Adding artificial viscosity coefficients to numerical modeling equations ends up with better results for problems with a small value of physical diffusion coefficient, i.e. less than $D_p=0.5$ (m²/s). For large amounts of physical diffusion, the physical diffusion would damp out unwanted numerical noises and stabilize the explicit solution procedure and the artificial viscosity terms are practically useless. In the absence of sharp gradient of desired variable (i.e. concentration of pollution), the second order artificial viscosity switches off. Therefore, the fourth order artificial viscosity ($\varepsilon_4$) would be the only effective artificial viscosity term for damping the oscillations.

Using $\varepsilon_4$ coefficient for diffusion problems can damp unwanted numerical errors for the cases with minor physical diffusion. However, using large values of the fourth order artificial viscosity may introduce some unwanted damping on spatial variation of desired variable (i.e. concentration) and reduce the peaks.

In the case that was studied in the present work, steady state current with no bed slope and bed friction is considered, and therefore, above mentioned conclusions are valid for such a case. But for real cases of concentration diffusion, the conflict between the artificial viscosity operators and time variation of current, turbulent modeling as well as bed slope and bed friction should be investigated. This may be subject of future works.

References:


