Abstract - The paper purpose is to dignify some aspects regarding the calculus model and technical solutions for multistage sounding rockets used to test spatial equipment and scientific measurements. The calculus methodology consists in numerical simulation of sounding rocket evolution for different start conditions. The rocket model presented will be with six DOF and variable mass. At this item, as novelty of the work we will use the rotation angles for describe the kinematical equations of the movement. The results analyzed will be the flight parameters and ballistic performances. The discussions area will focus around the technical possibility to realize sounding multi-stage rocket recycling military rocket engines.

Keywords— Multi-stage, Mathematic model, Sounding rocket, Simulation, Rotation, angle

NOMENCLATURE

\( \alpha \) - Attack angle (tangent definition);
\( \beta \) - Sideslip angle (tangent definition);
\( \beta_p \) - Azimuth angle;
\( \lambda_p \) - Geocentric latitude;
\( \psi \) - Azimuth angle;
\( \theta \) - Inclination angle;
\( \phi \) - Bank angle;
\( \rho \) - Air density;
\( \Omega \) - Body angular velocity;
\( \Omega_p \) - Earth spin:
\( A, B, C, E \) - Inertia moments;
\( C_{x}^{A}, C_{y}^{A}, C_{z}^{A} \) - Aerodynamic coefficients of force in the mobile frame;
\( C_{x}^{T}, C_{y}^{T}, C_{z}^{T} \) - Thrust coefficients in the mobile frame;
\( C_{x}^{T}, C_{y}^{T}, C_{z}^{T} \) - Thrust momentum coefficients in the mobile frame.

\( q_1, q_2, q_3, q_4 \) - Quaternion components;
\( \xi, \eta, \zeta \) - Rotation angles
\( F_0 = \frac{V^2}{2} S \) - Reference aerodynamic force;
\( H_0 = F_0 l \) - Reference aerodynamic couple;
\( T_0 \) - Reference thrust force
\( H_0' = T_0 l \) - Reference couple thrust
\( l \) - Reference length;
\( m \) - Mass;
\( m_i \) - Initial mass
\( m_f \) - Final mass
\( p, q, r \) - Angular velocity components along the axes of mobile frame;
\( S \) - Reference area;
\( T \) - Thrust;
\( I_Z \) - Total impulse;
\( t \) - Time;
\( V \) - Velocity;
\( u, v, w \) - Aircraft velocity components in a mobile frame;
\( V_x, V_y, V_z \) - Velocity components in Earth frame;
\( OX_p Y_p Z_p \) - Normal Earth-fixed frame [2];
\( Oxyz \) - body frame (mobile frame);
\( x_p, y_p, z_p \) - coordinates in Earth-fixed frame;
\( r \) - The distance between rocket and Earth center;
\( R_p \) - Earth radius.

I. INTRODUCTION

It is indisputable that today, the spatial program involves many collateral activities, like preliminary tests for equipment and the qualifications. It is well known that those auxiliary activities suppose huge technical and financial effort and this increases the total cost for any space program. Starting from this idea, the paper proposes an economical solution for the tests by using small multistage sounding rockets by recycling military rocket engines. This solution can be also used for quickly meteorological determination, taking into account that meteorological balloons are very slow. To solve these problems and in general for evaluating the launching
capability is necessary to elaborate a complex mathematical model that ensures the rigorous and accuracy evaluation of the flight data and ballistic parameters. The mathematical model presented below, developed with maximum of accuracy, seeks to answer these needs. To solve this we use the angles of rotation, which presents a number of advantages which will be outlined in section III of the paper. This model, allows us to evaluate two technical solutions, one of them based on three stage sounding rocket using the engines of short rocket 122 mm (Fig. 3), and the second using four boosters, also from short 122 mm rocket around a central body obtained from long rocket 122 mm (Fig. 4). These two technical solutions will be evaluated and the flight parameters and ballistic performances will be analyzed.

II. THE GRAVITY ACCELERATION, COMPLEMENTARY ACCELERATION, CONNECTION BETWEEN EARTH FRAME AND BODY FRAME

At the beginning we will start by analyzing the influence of the secondary parameters like the variation of the gravity acceleration with latitude and altitude and the influence of Earth spine about the sounding rocket trajectory.

The gravity acceleration and complementary acceleration

In order to write the movement equations we will use a geodesic frame [8] connected to the Earth. Due to diurnal spin, beside attraction force, we must consider two supplementary accelerations: carrying acceleration: \(-\mathbf{\Omega}_p \times \mathbf{V}\) and complementary acceleration (Coriolis acceleration) given by: \(-2\mathbf{\Omega}_p \times \mathbf{V}\), where Earth spin has the value: \(\mathbf{\Omega}_p = 7.2921 \times 10^{-5}\) s\(^{-1}\). If we designate \(r\) - the distance between rocket and Earth center:

\[
r = \sqrt{x_p^2 + (R_p + y_p)^2 + z_p^2},
\]

where: \(x_p; y_p; z_p\) are the rocket coordinates in the Earth frame and the Earth radius can be approximated by:

\[
R_p = a(1 - \alpha \sin^2 \lambda_p),
\]

the gravity acceleration components in the Earth frame are:

\[
g_{xp} = -g \frac{x_p}{r} - g_0 \frac{\Omega_p}{\mathbf{\Omega}} y_p; \quad g_{yp} = -g \frac{y_p + R_p}{r} - g_0 \frac{\Omega_p}{\mathbf{\Omega}} x_p;
\]

\[
g_{zp} = -g \frac{z_p}{r} - g_0 \frac{\Omega_p}{\mathbf{\Omega}},
\]

where the radials and polar components of the gravity acceleration [8], [3] are:

\[
g_r = g_0 - \mathbf{\Omega}_p \times r; \quad g_0 = g_0 \frac{\Omega_p}{\mathbf{\Omega}} \sin \lambda_p,
\]

and \(\Omega_p; \Omega_{xp}; \Omega_{yp}; \Omega_{zp}\) - the spin components are given by:

\[
\Omega_{xp} = \Omega_p \cos \lambda_p \cos \beta_p; \quad \Omega_{yp} = \Omega_p \sin \lambda_p;
\]

\[
\Omega_{zp} = -\Omega_p \cos \lambda_p \sin \beta_p,
\]

where the two angels used are: \(\beta_p\) - azimuth angle and \(\lambda_p\) - geocentric latitude.

On another hand, complementary acceleration is:

\[
a_c = -2\mathbf{\Omega}_p \times \mathbf{V},
\]

with the Earth frame components given by:

\[
a_{exp} = 2(V_{xp} \Omega_p - V_{yp} \Omega_p^2); \quad a_{exp} = 2(V_{yp} \Omega_p - V_{xp} \Omega_p^2);
\]

\[
a_{exp} = 2(V_{yp} \Omega_p - V_{xp} \Omega_p^2),
\]

where \(V_{xp}; V_{yp}; V_{zp}\) are the Earth frame velocity components.

The connection between Earth frame and body frame

The Earth frame is a geocentric frame with the origin in the mass center of the Earth, which can be considerate an ellipsoid of revolution [2],[3], [8]. In order to obtain body frame we are passing through three intermediary frames. The first frame is the starting frame \((Ox,y,z)\), which has \(y\) axis normal to the tangent plane at the Earth’s surface (Fig.1). Booths frames participant in Earth rotation are not inertial frames. But, if we introduce as corrections the Earth spin influence by gravity and complementary acceleration, we can consider them as inertial frame.

\[
[\lambda_p = \lambda_g - \gamma_p],\quad (7)
\]

the connection between frames is:

\[
[x, y, z] = A \gamma_p [x_p, y_p, z_p].\quad (8)
\]

Overlapping the Earth frame above starting frame can be done by the rotation matrix:

\[
A \gamma_p = \begin{bmatrix}
sb^2 + cb^2 \cdot cg & -sb \cdot cb(1 - cg) & -cb \cdot sg \\
-cb \cdot sg & sb^2 + cb^2 \cdot cg & sb \cdot sg \\
-cb \cdot sg & sb \cdot sg & cg
\end{bmatrix},\quad (9)
\]

where:

\[
sb = \sin \beta_p; \quad cb = \cos \beta_p; \quad sg = \sin \gamma_p; \quad cg = \cos \gamma_p.
\]

In order to obtain the angle \(\gamma_p\) between geocentric and geodetic normal we must start from the relation [8]:

\[
a^2 \tan \beta_p = b^2 \tan \lambda_g,\quad (10)
\]
where $a$, $b$ are the Earth semi-axis. From relation: $\tan \gamma_p = \frac{\alpha(1-\alpha/2)\sin 2\lambda_p}{1-2\alpha(1-\alpha/2)\cos^2 \lambda_p}$. (11)

Obviously, if the start frame origin is on the Equator or on North or South Pole the angle $\gamma_p$ is null and the matrix $A_p$ became unitary matrix.

The next intermediary frame is the initial starting frame ($O_0X_0Y_0Z_0$), which overlap the starting frame in the launching moment. The starting frame which is attached to the Earth, is rotating around the polar axe related to the initial starting frame, considerate fix, with an angle equal with rotation angle of the Earth in this time.

\[ \gamma_0(t) = \frac{1}{180} \cdot \pi \cdot \Omega \cdot t. \] (12)

The connection between these frames is given by:
\[ X_0 \ Y_0 \ Z_0 = A_{dp}[x \ y \ z]^T. \] (13)

where the rotation matrix elements are:
\[ a_{11} = \cos^2 \beta_p \cos^2 \lambda_g (1 - \cos \Omega_p t) + \cos \Omega_p t; \]
\[ a_{12} = \sin \beta_p \cos \beta_p \cos \lambda_g (1 - \cos \Omega_p t) - \sin \lambda_g \sin \Omega_p t; \]
\[ a_{13} = \cos \beta_p \sin \lambda_g \cos \lambda_g (1 - \cos \Omega_p t) + \sin \beta_p \cos \lambda_g \sin \Omega_p t; \]
\[ a_{21} = \sin \beta_p \cos \beta_p \cos \lambda_g (1 - \cos \Omega_p t) + \sin \lambda_g \sin \Omega_p t; \]
\[ a_{22} = \sin^2 \beta_p \cos^2 \lambda_g (1 - \cos \Omega_p t) + \cos \Omega_p t; \]
\[ a_{23} = \sin \beta_p \sin \lambda_g \cos \lambda_g (1 - \cos \Omega_p t) - \cos \beta_p \cos \lambda_g \sin \Omega_p t; \]
\[ a_{31} = \sin \beta_p \cos \beta_p \cos \lambda_g (1 - \cos \Omega_p t) - \sin \lambda_g \sin \Omega_p t; \]
\[ a_{32} = \sin \beta_p \sin \lambda_g \cos \lambda_g (1 - \cos \Omega_p t) + \cos \beta_p \cos \lambda_g \sin \Omega_p t; \]
\[ a_{33} = \sin^2 \lambda_g (1 - \cos \Omega_p t) + \cos \Omega_p t. \] (14)

Obviously, if the flight time is short, the matrix $A_{dp}$ becomes unitary matrix.

Finally, with Euler angle $(\psi, \theta, \phi)$ or Euler modified angle $(\psi', \psi'', \psi''')$ or Hamilton’s quaternion $(q_1, q_2, q_3, q_4)$ or rotation angles $\xi, \eta, \zeta$ [10], we can overlap the initial start frame with the body frame. The rotation matrix $A_1$, which makes this transformation, will be shown afterwards.

In the issue, in order to pass some elements from Earth frame to the body frame we need three rotations, which can be concentrated in a single matrix:
\[ A_p = A_1 A_{dp} A_m. \] (15)

which is the rotation matrix between Earth frame and body frame.

### III. GENERAL MOVEMENT EQUATIONS

**Kinematical equations**

Unlike paper [4], which covers the regular ballistic Rockets, where the kinematical equations used Euler angles, in our case, including launching rocket for the satellite, with the almost vertical initial launching direction, the papers [3], [5], [6], [7], [8] recommend modified Euler angles, which have first rotation in vertical plane. Using kinematical equations written with Euler angles, in addition to benefits related to the significance of physical measurable sizes, the following drawback is involved: the use of trigonometric functions and difficulties that these place in the building program algorithms.

Therefore, in some works [3], it is proposed to build rotation matrices and a connection matrix related to rotation speed by using an algebraic operator called Hamilton's quaternion. Quaternion of Hamilton is an operator that expresses the rotation. If the rotation is done in connection with the angle $\sigma$ around an axis with the unit vector:
\[ e_\sigma = \mathbf{i} + \mathbf{j} + \mathbf{k}, \] (16)

the sizes:
\[ q_1 = l \sin \frac{\sigma}{2}; \quad q_2 = m \sin \frac{\sigma}{2}; \quad q_3 = n \sin \frac{\sigma}{2}; \quad q_4 = \cos \frac{\sigma}{2} \] (17)

are called quaternion components.

Paper [10] were first introduced a group of three angles, called the rotation angles, the size beings used to describe the movement of aircraft. As part of this work will resume this problem, aiming at the use of these sizes to write movement equations for sounding rocket.

Angles of rotation are a derivative form of quaternion, which has the advantage that it can be measured easily on board of the aircraft or rocket. They retain the advantages of quaternion, removing singularity from kinematical equations written with the attitude angles. Also, allow the polynomial expression of the kinematical equations, important advantage in building high-speed algorithms and easily implemented on hardware support. Angles of rotation retain the advantage of angles Euler type, that of being a quantities directly measurable with a concrete physical meaning.

It is well known that a sequence of rotations of a rigid body with a fixed point can be replaced by a single rotation $\sigma$ around an axis through the fixed point heaving unit vector indicate by (15). Thus, the overall rotation angle can be expressed by the superposition of three simultaneous rotations along the mobile frame axes. The sizes:
\[ \xi = \sigma t; \quad \eta = \sigma m; \quad \zeta = \sigma n \] (18)

are called rotation angles:
\[ \xi \quad \text{- Rotation angle around roll axis (roll angle)}; \]
η - Rotation angle around pitch axis (pitch angle);
ζ - Rotation angle around yaw axis (yaw angle).

where, from relation (16) we have:

\[ l = \frac{q_1}{\sqrt{1-q_1^2}}; \quad m = \frac{q_2}{\sqrt{1-q_2^2}}; \quad n = \frac{q_3}{\sqrt{1-q_3^2}}; \quad \sigma = 2 \arccos(q_4). \] (18)

The relations between the rotation angles and quaternion components are:

\[ \xi = eq_1; \quad \eta = eq_2; \quad \zeta = eq_3, \] (19)

where:

\[ \sigma = \sqrt{\xi^2 + \eta^2 + \zeta^2}; \quad e = \frac{2\arccos(q_4)}{\sqrt{1-q_4^2}} = \frac{\sigma}{\sin(\sigma/2)}. \] (20)

Using rotation angles work [3] show that the rotation matrix has the following form:

\[
A_i = \begin{bmatrix}
a\xi^2 + c & -a\eta\xi - b\zeta - a\zeta\xi + b\eta \\
-a\xi\eta - b\zeta & -a\eta^2 - c - a\zeta\eta - b\xi \\
-a\zeta\xi + b\zeta & -a\zeta\eta + b\xi & -a\xi^2 - c
\end{bmatrix}
\] (21)

where:

\[ a = \frac{1-c}{\sigma^2}; \quad b = \frac{\zeta}{\sigma}; \quad c = \cos \sigma; \quad s = \sin \sigma \] (22)

Using the inverse of this matrix

\[ B_i = A_i^{-1} = A_i^T \] (23)

we can obtain the components of acceleration in the Earth frame from components of acceleration in the body frame, used in the dynamical equations.

Like the kinematical equations we can write the relation:

\[
\begin{bmatrix}
\dot{x}_p \\
\dot{y}_p \\
\dot{z}_p
\end{bmatrix} = \begin{bmatrix}
V_{xp} \\
V_{yp} \\
V_{zp}
\end{bmatrix}, \tag{24}
\]

where \( V_{xp}, V_{yp}, V_{zp} \) are components of the velocity in the Earth frame.

In order to obtain the connection between the derivatives of rotation angles and components of rotation velocity in the body frame we use the following relation indicated in [3]:

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\eta} \\
\dot{\zeta}
\end{bmatrix} = \begin{bmatrix}
f_\xi + h f\eta + \xi/2 \\
f_\xi + h f\eta + \zeta/2 \\
f_\xi + h f\eta + \zeta/2
\end{bmatrix}
\] (25)

which, together with the relation (24) represent kinematical equation.

Because the rotation matrix (21) is the same regardless of the variables used, in work [3] are obtained the following relationships between different variables (Euler angles, rotation angles):

\[
\tan \Phi = \frac{a_{i,5}}{a_{i,3}} = \frac{a\zeta\eta + b\zeta}{a\zeta^2 + c}; \quad \tan \Theta = \frac{a_{i,5}}{a_{i,3} \cos \Psi} = \frac{a\zeta\eta - b\eta}{a\zeta^2 + c}; \tag{26}
\]

which together with (17) are used for transformation between rotation angles, quaternion components and Euler angles.

**Dynamical equations**

Developing cross products from paper [3], with the supplementary notations from [3], [4] we can obtain matrix representation of the dynamical equations:

- Force equations in the Earth frame

\[
\begin{bmatrix}
\dot{V}_{xp} \\
\dot{V}_{yp} \\
\dot{V}_{zp}
\end{bmatrix} = \begin{bmatrix}
B_p \quad F_0 \quad C_y \quad C_z \\
 \quad T_0 \quad C_y' \quad C_z' \\
 \quad 0 \quad C_y' \quad C_z'
\end{bmatrix} + \begin{bmatrix}
g_{xp} \\
g_{yp} \\
g_{zp}
\end{bmatrix} + \begin{bmatrix}
a_{xp} \\
a_{yp} \\
a_{zp}
\end{bmatrix}; \tag{27}
\]

- Moment equations in the body frame

\[
\begin{bmatrix}
\dot{\rho} \\
\dot{\eta} \\
\dot{\zeta}
\end{bmatrix} = J^{-1} \begin{bmatrix}
(B - C q \eta r) \quad (C - A p) \eta \eta \quad (A - B p) \eta \zeta
\end{bmatrix}, \tag{28}
\]

where we denoted:

\[ J^{-1} = \begin{bmatrix}
1/A & 0 & 0 \\
0 & 1/B & 0 \\
0 & 0 & 1/C
\end{bmatrix}. \] (29)

The inertial moment inverse matrix, where the inertial moments are given by:

\[ A = \int (y^2 + z^2) \, dm; \quad B = \int (z^2 + x^2) \, dm; \quad C = \int (x^2 + y^2) \, dm. \] (30)

and the matrix \( B_p = A_p^T \) is the transpose matrix given by (23).

For the aerodynamic coefficients calculus we used the method indicated in [4] and [9], based on polynomial series expanding:

\[
C_i = a_i + a_2(\alpha^2 + \beta^2) + a_3(\alpha^4 + \beta^4) + a_4\alpha^2\beta^2 + a_1(\dot{\tilde{z}} - \dot{z}); \tag{31}
\]

where, by definition [9]

\[ \alpha = -\arctan(v/u), \quad \beta = \arctan(w/u). \] (32)

The following notations are used for the non dimensional angular velocities:

\[ \dot{\rho} = \rho l/V; \quad \dot{\eta} = q l/V; \quad \dot{\zeta} = r l/V. \] (33)

Also, we have the thrust coefficients:

\[ C_i^T; C_y^T; C_z^T; C_m^T; C_n^T. \] (34)

**IV. INPUT DATA, CALCULUS ALGORITHM AND RESULTS**

**Input data for the model**

Figure 3 shows the first model, called „VLS T3”, and main characteristics are included in Table 1.
Fig. 3 VLS T3 configuration

Fig. 4 shows the second model, called „VLS B4”, and main characteristics are included in Table II.

**Test calculus**

For the test calculus the following initial conditions were used:

- Geographic orientation:
  - Azimuth angle $\beta_p = 90^\circ$ (towards the East);
  - Geocentric latitude $\lambda_p = 45^\circ$ (Romania latitude);
  - Altitude: $y_o = 1[m]$

Initial velocity $V_0 = 40[m/s]$;
Initial inclination angle $\theta_0 = 85^\circ$

**Results**

In the figures 5-8 are comparatively shown the flight parameters and the ballistic performances of the two models.

Fig. 5 shows the mass variation in time along the trajectory.

We can see a quickly mass decrease, followed by constant period, after burnout. In Fig 6 is presented the velocity diagram. It can be observed the difference between the models, the velocity of the second being greater.

Consequently, due to the velocity difference the trajectory of the second is higher (Fig. 7). Finally, in Fig. 8 we have
shown the inclination angle in time. It can be observed that at trajectory apex we have greatness instability especially for the second model which attend higher trajectory.

![Fig. 7 Ballistic trajectory](image1)

![Fig. 8 Inclination angle diagram](image2)

**V. CONCLUSIONS**

The paper presents synthesis aspects of the simulation model, developed for the calculation of an operational rocket - which will launch equipment which will be tested with the aim of them integrate into complex spatial systems. The application is made for two variants of the sounding rockets, whence will be tested in the national projects. We considered more possible solutions and we presented and analyzed the flight parameters for two sounding rockets that can be used. Since the major objective of the sounding rockets consists of testing solution for assembling and detachment of multi-stage rockets and to launch at large angles, to eliminate the additional risk, it will be used the rocket motors from current production connected in parallel or in tandem, solution what was presented during the work. In conclusion, the main sub-assemblies of the sounding rockets, will be provided from current production, the engine being developed in Romania and being in significant amounts in deposits. Assembly and testing of the ground will be made at the plant where usually such systems are manufactured, and finally the sounding rockets will be tested by firing in an area from the Black Sea maybe next year.

**REFERENCES**