Absence of eigenvalues for integro-differential operators

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Abstract: - In this paper we obtain the sufficient conditions (but in some sense even the optimum) for the absence of eigenvalues of the operators (in general nonselfadjoints) generated by integro-differential expressions. The main results are obtained using an abstract scheme.

Keywords: - Spectral theory, non-selfadjoint operators, relatively bounded perturbations, differential, and integro-differential operators, eigenvalue.

1 Introduction

The problem of the absence of eigenvalues of integro-differential operators just as other ones involving spectral properties of an integro-differential operator arose from the practical necessities of plasma oscillations theory (in this respect we note the works of D.Bohm and E. Grose [1], N. G. Van Kampen [2] and K. M. Case [3]), of mathematical theory of scattering of neutrons (see, for instance, J. Lehner and G. M. Wing [4] and see also [5]) and of other principle situation from quantum physics and mechanics. We also note the works [6-8] (as well as the references therein) in which mathematical models involving integro-differential operators can be found as well.

A part of the results of the present paper was announced without any proof in our article [9].

The Hilbert spaces are denoted by $\mathrm{H}, \mathrm{H}_1, \ldots$ the inner products and the norms in those Hilbert spaces are denoted by $(.,.)_{\mathrm{H}}$ and $\| \ \|_{\mathrm{H}}$. The set of linear operators closed and densely defined on H_1 with values in H_2 is denoted by $C(\mathrm{H}_1, \mathrm{H}_2)$. $B(\mathrm{H}_1, \mathrm{H}_2)$ stands for the Banach space of all bounded linear operators defined on H_1 with values in H_2 and $B_\infty(\mathrm{H}_1, \mathrm{H}_2)$ the subspace of $B(\mathrm{H}_1, \mathrm{H}_2)$ consisting of all compact operators defined on H_1 with values in H_2. For every operator A in H, the domain, the range, the resolvent set and the spectrum are denoted by $D(A)$, $R(A)$, $\rho(A)$ and $\sigma(A)$, respectively. The point spectrum of the operator A (the set of all eigenvalues of A) is denoted by $\sigma_p(A)$.

2 The absence of eigenvalues of some integro-differential operators

These results are similarly with the results which refer to the differential operators and with the other results which refer to the Wiener-Hopf-type operators (see [10]).

2.1. We realize the following scheme (see [11]).

Let H be an operator defined on the Hilbert space H, of the form

$$H = A + B, \quad (1)$$

where B has the form $B = \sum_{\alpha,\beta=0}^{n} S_\alpha T_{\alpha\beta} R_\beta$,

and the operators $A, S_\alpha, R_\beta, T_{\alpha\beta}$ $(\alpha, \beta = 0, \ldots, n)$ satisfy the following conditions:

(i) A is closed and densely defined;

(ii) the complex number λ is not an eigenvalue of the operator A, that is $\lambda \notin \sigma_p(A)$;

(iii) the operators $S_\alpha, R_\beta, T_{\alpha\beta}$ $(\alpha, \beta = 0, \ldots, n)$ act in the H space with the properties

$$D(A) \subset D(S_\alpha) \cap D(R_\beta),$$
$$T_{\alpha\beta} \in B(\mathrm{H}) \, (\alpha, \beta = 0, \ldots, n).$$

As well as in the paper [11], in the H space we consider the operators family L_τ $(\tau \geq 0)$, with property:

(iv) $\ker L_\tau = 0 \left(\tau \in R_+\right)$ and $L_0 = I$ (I is the identical operator in the H space). On the domain $D_\tau = D(L_\tau)$ of the operator L_τ $(\tau \geq 0)$ we define the norm $\|u\|_\tau = \|L_\tau u\|_{\mathrm{H}}$ $(u \in D_\tau)$.

Such as result we obtain the normalized space H_τ (in general uncomplete).

Clearly $\mathrm{H}_0 = \mathrm{H}$. Moreover, we suppose that the following properties are fulfilled:

(v) there is $\tau \geq 0$ such that if $v_\beta \in \mathrm{H}$ and $S_\alpha T_{\alpha\beta} R_\beta \in R(A - \lambda I)$ $(\alpha, \beta = 0,...,n)$, then $v_\beta \in \mathrm{H}_\tau$,

$$R_\gamma (A - \lambda I)^{-1} S_\alpha T_{\alpha\beta} R_\beta v_\beta \in \mathrm{H}_\tau$$

and

$$\left\|R_\gamma (A - \lambda I)^{-1} S_\alpha T_{\alpha\beta} v_\beta\right\|_\tau \leq a\|v_\beta\|_\tau ;$$

$$(0 < a < 1;\ \alpha, \beta, \gamma = 0,...,n)$$

(vi) if $v_\beta = R_\beta u$, where

$$u + \sum_{\alpha,\beta=0}^{n} (A - \lambda I)^{-1} S_\alpha T_{\alpha\beta} R_\beta u = 0,$$

then

$$\left\|R_\gamma (A - \lambda I)^{-1} S_\alpha T_{\alpha\beta} v_\beta\right\|_\tau \leq c \cdot \|v_\beta\|_{\tau'}$$

$$(c = const.;\ \tau \geq \tau';\ \alpha, \beta, \gamma = 0,...,n).$$

Remark. Throughout this paper we consider only the situation on which the perturbation operator B is subordinated to unperturbed operator A. In this connection we assume additionally that the function $q_{\alpha\beta}$ and, respectively the kernels $k_{\alpha\beta}(x, y)$ either for $\alpha = n$ or $\beta = n$ are identical equal to zero. Thus in the sum determining B either $\alpha = 0,...,n$ and $\beta = 0,...,n-1$ or $\alpha = 0,...,n-1$ and $\beta = 0,...,n$.

Lemma 1. Let be the operator H of the form (2). If the conditions (i)-(vi) are fulfilled, then λ is not an eigenvalue of the operator H.

Proof. Suppose on the contrary, let λ be an eigenvalue of the operator H, then there is $u \neq 0, u \in \mathrm{H}$, such that

$$Hu = \lambda u . \quad (2)$$

Let H be a Hilbert space formed like a direct sum from $(n+1)$ spaces H, namely

$$\widetilde{\mathrm{H}} = \sum_{k=0}^{n} \oplus \mathrm{H} .$$

We consider the operator $\widetilde{R} : \mathrm{H} \to \widetilde{\mathrm{H}}$ such as

$$\widetilde{R} = (R_\alpha)_{\alpha=0}^{n} \quad \left(u \in \bigcap\nolimits_{\alpha=0}^{n} D(R_\alpha)\right),$$

the operator $\widetilde{T} : \widetilde{\mathrm{H}} \to \widetilde{\mathrm{H}}$,

$$\widetilde{T}\widetilde{v} = \left(\sum_{\beta=0}^{n} T_{\alpha\beta} v_\beta\right)_{\alpha=0}^{n} \quad \left(\widetilde{v} = (v_\beta)_{\beta=0}^{n} \in \widetilde{\mathrm{H}}\right)$$

and we denotes the operator $\widetilde{S} : \widetilde{\mathrm{H}} \to \mathrm{H}$ thus

$$\widetilde{S}\widetilde{v} = \sum_{\alpha=0}^{n} S_\alpha v_\alpha \left(\widetilde{v} = (v_\alpha)_{\alpha=0}^{n} \in \widetilde{\mathrm{H}}, v_\alpha \in D(S_\alpha); \alpha = 0,...,n\right).$$

We remark that $\widetilde{S}\widetilde{T}\widetilde{R} = \sum_{\alpha,\beta=0}^{n} S_\alpha T_{\alpha\beta} R_\beta$ and in accordance with (3) it results that

$$Au + \widetilde{S}\widetilde{T}\widetilde{R}u = \lambda u,$$

or

$$u + (A - \lambda I)^{-1} \widetilde{S}\widetilde{T}\widetilde{R}u = 0. \quad (3)$$

We denote $\widetilde{v} = \widetilde{R}u$. Because $\lambda \notin \sigma_p(A)$ it results that $\widetilde{v} \neq 0$.

If $\widetilde{v} = 0$ then $R_\alpha u = 0$ $(\alpha = 0,...,n)$ and we obtain the contradiction $Au = \lambda u$.

Because $D(A) \subset D(R_\alpha)$ $(\alpha = 0,...,n)$ on the basis of (4) it results that $u \in D(\widetilde{R})$ and we obtain

$$\widetilde{v} + \widetilde{R}(A - \lambda I)^{-1} \widetilde{S}\widetilde{T}v = 0 . \quad (4)$$

The family of the operators $(L_\tau)_{\tau\geq 0}$ from the space H is in correspondence with the family of the operators $(L_\tau)_{\tau\geq 0}$ from the space $\widetilde{\mathrm{H}}$, where

$$\widetilde{L}_\tau \widetilde{v} = (L_\tau v_\alpha)_{\alpha=0}^{n}$$

$$\left(\widetilde{v} = (v_\alpha)_{\alpha=0}^{n};\ v_\alpha \in D_\tau;\ \tau \geq 0;\ \alpha = 0,...,n\right).$$

In accordance with condition (v), the condition (5) involves the following relation

$$|\widetilde{v}|_\tau = \left\|\widetilde{L}_\tau \widetilde{v}\right\| = \left\|\widetilde{L}_\tau \widetilde{R}(A - \lambda I)^{-1} \widetilde{S}\widetilde{T}\widetilde{v}\right\| \leq a|v_\beta|_\tau$$

$$(0 < a < 1;\ \tau \geq 0).$$

Therefore

$$|\widetilde{v}|_\tau \leq a|\widetilde{v}|_\tau \ (0 < a < 1;\ \tau \geq 0). \quad (5)$$

Previously we mentioned that $\widetilde{v} \neq 0$, and on the base of (vi) it results that $|\widetilde{v}|_\tau < \infty$.

Because on the base of (6) it results $a \geq 1$, we obtain a contradiction with $0 < a < 1$. Thus, the lemma is proof.

2.2. The integro-differential operator who's the spectrum is studied in this paper, has the form

$$H = \sum_{\alpha,\beta=0}^{n} D^{*\alpha} M_{\alpha\beta} D^{\beta}, \quad (6)$$

where $M_{\alpha\beta}$ $(\alpha,\beta = 0,...,n)$ are of the type

$$(M_{\alpha\beta} u)(x) = a_{\alpha\beta} u(x) + q_{\alpha\beta}(x)u(x) + \\ + \int_{R_+} k_{\alpha\beta}(x,y)u(y)dy$$

and $(\alpha,\beta = 0,...,n;\ x \in R_+)$, and they act in the space $L_2(R_+)$.

We consider the operator H with its maximal domain of definition, i.e. with the domain consisting of all functions $u \in W_p^n(R_+)$ ($W_p^n(R_+)$ denotes the Sobolev space of order n over R_+) such that $D^{\beta}u$ belong to the domain of $M_{\alpha\beta}$ for each $\alpha,\beta = 0,...,n$, and $M_{\alpha\beta}D^{\beta} \in W_p^n(R_+)$.

In particular, if $q_{\alpha\beta}$ $(\alpha,\beta = 0,...,n)$ are continuous and bounded functions together with their derivates of the $(n-1)$ order on the semi-axis R_+ and the kernels $k_{\alpha\beta}(x,y)(\alpha,\beta = 0,...,n)$ are such that the integral operators with kernels $\frac{\partial^j k_{\alpha\beta}(x,y)}{\partial x_j}$ $(j = 0,...,\alpha;\ \alpha,\beta = 0,...,n)$ are bounded on $L_p(R_+)$, then the domain of H is considered to be the Sobolev space $W_p^{2n}(R_+)$.

Here $a_{\alpha\beta}$ $(\alpha,\beta = 0,...,n;\ a_{nn} \neq 0)$ are complex numbers, $q_{\alpha\beta}$ $(\alpha,\beta = 0,...,n)$ and, respectively, the kernels $k_{\alpha\beta}(x,y)(\alpha,\beta = 0,...,n)$ are smooth as it will be necessary complex-valued functions.

D denotes the differential operator $D = i\frac{d}{dx}$ with the domain of definition determined by the set of all functions $u \in L_p(R_+)$ which are absolutely continuous on every bounded interval of the positive semi-axis and $u' \in L_p(R_+)$. More, $u(0) = 0$.

It supposes that the operator H acts in the space $L_2(R_+)$.

Let consider the operator

$$(B_{\alpha\beta}u)(x) = q_{\alpha\beta}(x)u(x) + \\ + \int_{R_+} k_{\alpha\beta}(x,y)u(y)dy \ (\alpha,\beta = 0,...,n)$$

and then the operator H is represented like a perturbed operator $H = A + B$ where

$$A = \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta} D^{*\alpha} D^{\beta}$$

and

$$B = \sum_{\alpha,\beta=0}^{n} D^{*\alpha} B_{\alpha\beta} D^{\beta}.$$

It is known that $D \subset D^*$, the resolvent set $\rho(D^*)$ of the operator D^* coincides with the open upper half-plain and so the spectrum of the operator D^* consists of all points of the closed lower half-plain. Moreover, the point spectrum $\sigma_p(D^*)$ on the real axis is absent.

Spectral properties of the operator A are well known and, in particular, information on the spectrum of A can be derived, for instance from [12]. Here, let us only make some remarks that will be necessary for our further discussions. Let λ be a complex number and denote by ξ_k $(k = 1,...,2n)$ the roots of the polynomial $A(\xi) - \lambda$. Then

$$A - \lambda I = a_{nn} \prod_{j=1}^{2n} (D - \xi_k I). \quad (7)$$

The resolvent set of the operator D coincides with the open upper half-plane $\prod_+ = \{z \in C / \Im z > 0\}$, the open lower half-plane $\prod_-$ is filling with the point spectrum of D and the real axis being in the continuous part of the spectrum is free from the point spectrum of A, i.e. $R \subset \sigma_c(D)$ and $R \cap \sigma_p(D) = \Phi$. We note that for $\Im z > 0$ one has

$$(D - zI)^{-1} u(x) = -i \int_x^{\infty} \exp(iz(y - x))u(y)dy. \quad (8)$$

The representation (8) for $\Im z = 0$ is also holds, but then it is necessary to consider u from the range of the operator $D - zI$. Now, from the equality (7) it is easy to conclude that $\lambda \in \sigma(A)$ if and only if the roots of the polynomial $A(\xi) - \lambda$ are contained in the closed lower half-plane $\Im \xi \le 0$. Moreover, a point λ is not an eigenvalue of A provided that the polynomial $A(\xi) - \lambda$ has its zeros only in $\Im \xi \ge 0$.

Next, let λ be not an eigenvalue of the unperturbed operator A. According to what has already been said, the polynomial $A(\xi)-\lambda$ can be represented as follows

$$A(\xi)-\lambda=\prod_{j=0}^{r}\left(\xi-\xi_j\right)^{m_j}A_1(\xi), \quad (9)$$

where ξ_j $(j=0,...,r)$ are real pair wise distinct numbers, $m_j \geq 0$ $(j=0,...,r)$ and $A_1(\xi)$ is a polynomial, the roots of which belong to the upper half-plane $\Im\xi>0$. We let $m_j=0$ for the case in which the polynomial $A(\xi)-\lambda$ has no zeros on the real axis.

Theorem 1. Let H be an operator of type (1) which acts in the space $L_2(R_+)$, the complex number λ is not an eigenvalue of the operator A, the polynomial $A(\xi)-\lambda$ can be represent of the form (8), $m=\max\{m_j / j=0,...,r\}$.

If

$$(1+x)^{\delta} q_{\alpha\beta}(x) \in L_{\infty}(R_+)$$
$$(\delta>m;\ \alpha,\beta=0,...,n-1)$$

and if $k_{\alpha\beta}(x,y)=0$ for $x>y$, and the integral operators with kernels

$$(1+x)^{\delta} k_{\alpha\beta}(x,y)\,(\alpha,\beta=0,...,n-1;\ \delta>m)$$

are bounded in $L_2(R_+)$, then λ is not an eigenvalue of the operator H.

Proof. We verify the conditions (i)-(vi). The Hilbert space $\widetilde{\mathrm{H}}$ is formed like a direct sum from $(n+1)$ spaces $L_2(R_+)$,

$$\widetilde{\mathrm{H}}=\sum_{k=0}^{n}\oplus L_2(R_+).$$

We denote by $R_\beta=D^\beta$, $S_\alpha=D^{*\alpha}$ and $T_{\alpha\beta}$ is considered equal with $B_{\alpha\beta}$ $(\alpha,\beta=0,...,n)$.

Let us consider the operators

$$(L_\tau u)(x)=(1+x)^\tau u(x)$$
$$(u\in L_2(R_+)\cap Dom(L_\tau),\ \tau\geq 0).$$

Clearly the conditions (i)-(iv) are fulfilled and remain to verify the conditions (v) and (vi).

Because the operator

$$D^\gamma (A-\lambda I)^{-1} D^{*\alpha}\ (\alpha,\gamma=0,...,n)$$

on his definition domain is a linear combination of the operators of the type $(D^*-\mu I)^{-l}$ (where $l=1,...,m$ and $\Im\mu\geq 0$), it is sufficient to estimate the norm

$$\left\|(D^*-\mu I)^{-l}B_{\alpha\beta}v_\beta\right\|_\tau$$
$$(v_\beta\in L_2(R_+);\ \alpha,\beta=0,...,n;\ l=1,...,m;\ \tau\geq 0).$$

For this it uses the following Hardy inequality (see [13])

$$\|u\|_\tau \leq c(\tau)\left\|(D^*-\mu I)u\right\|_{\tau+\delta}\ (\delta>1), \quad (10)$$

where $c(\tau)\to 0$, when $\tau\to\infty$. This inequality is applied of l times successively.

We have two cases: $\Im\mu=0$ and $\Im\mu>0$.

In the involved case $\Im\mu=0$, in accordance with previous inequality (see the previous Hardy inequality (9)) it obtains

$$\left\|L_{\tau+j}(D^*-\mu I)^{-1}L^{-1}_{\tau+j+1}\right\|\leq a(\tau)\,(j\in N),$$

where $a(\tau)\to 0$, when $\tau\to\infty$ (see lemma 2 of [14]).

Therefore

$$\left\|(D^*-\mu I)^{-m}B_{\alpha\beta}v_\beta\right\|_\tau \leq a(\tau)\left\|L_{\tau+m}B_{\alpha\beta}L_\tau^{-1}\right\|\cdot\left\|v_\beta\right\|_\tau,$$
$$(\alpha,\beta=0,...,n;\ \tau\geq 0)$$

where $a(\tau)\to 0$, when $\tau\to\infty$.

In concordance with the conditions imposed of the functions $q_{\alpha\beta}$ and $k_{\alpha\beta}$ $(\alpha,\beta=0,...,n)$ it is obtain $\left\|L_{\tau+m}B_{\alpha\beta}L_\tau^{-1}\right\|\leq c$

(c is the constant which does not depend on τ; $\alpha,\beta=0,...,n;\ \tau\geq 0;\ \delta>m$).

In conclusion $\left\|(D^*-\mu I)^{-m}B_{\alpha\beta}v_\beta\right\|_\tau \leq a(\tau)\left\|v_\beta\right\|_\tau$ $(\alpha,\beta=0,...,n;\tau\geq 0)$, where $a(\tau)\to 0$, when $\tau\to\infty$.

If $\Im\mu>0$, in concordance with lemma 1 of [14] it is true that

$$\left\|(D^*-\mu I)^{-l}B_{\alpha\beta}v_\beta\right\|_\tau \leq a(\tau)\left\|L_{\tau+\varepsilon}B_{\alpha\beta}L_\tau^{-1}\right\|\cdot\left\|v_\beta\right\|_\tau$$
$$(\alpha,\beta=0,...,n;\ \tau\geq 0;\ \varepsilon>0),$$

where $a(\tau)\to 0$, when $\tau\to\infty$.

Thus, we obtain

$$\left\|L_{\tau+\varepsilon}B_{\alpha\beta}L_\tau^{-1}\right\|\leq c\ (\alpha,\beta=0,...,n;\ \tau\geq 0;\ \varepsilon>0),$$

where c is the constant which does not depend on τ; $\alpha,\beta=0,...,n;\ \tau\geq 0;\ \varepsilon>0$.

Therefore

$$\left\|(D^*-\mu I)^{-l}B_{\alpha\beta}v_\beta\right\|_\tau \leq a(\tau)\left\|v_\beta\right\|_\tau$$
$$(\alpha,\beta=0,...,n;\ \tau\geq 0;\ \varepsilon>0),$$

where $a(\tau) \to 0$, when $\tau \to \infty$.

The property (v) is proof for both of cases.

For proof of the condition (vi) we stand the same situations as well as in property (v).

If $\Im\mu = 0$ it obtains

$$\left\| (D^* - \mu I)^{-m} B_{\alpha\beta} v_\beta \right\|_\tau \le a(\tau) \left\| L_{\tau+m} B_{\alpha\beta} L_{\tau-\varepsilon}^{-1} \right\| \cdot \left\| v_\beta \right\|_{\tau-\varepsilon}$$

$$(\alpha, \beta = 0,\dots,n;\ \tau \ge \varepsilon;\ \varepsilon > 0),$$

where $a(\tau) \to 0$, when $\tau \to \infty$.

Thus $\left\| L_{\tau+m} B_{\alpha\beta} L_{\tau-\varepsilon}^{-1} \right\| \le c$

($\alpha, \beta = 0,\dots,n;\ \tau \ge \varepsilon;\ \delta > m$, c is the constant which does not depend on τ), and therefore

$$\left\| (D^* - \mu I)^{-m} B_{\alpha\beta} v_\beta \right\|_\tau \le c \left\| v_\beta \right\|_{\tau-\varepsilon}$$

($\alpha, \beta = 0,\dots,n;\ \tau \ge \varepsilon;\ \delta > m;\ c$ is constant).

If $\Im\mu > 0$ then $\left\| L_\tau (D^* - \mu I)^{-1} L_\tau^{-1} \right\| \le a(\tau),$

where $a(\tau) \to 0$, when $\tau \to \infty$ (see lemma 1 of [4]).

Therefore

$$\left\| L_\tau B_{\alpha\beta} L_{\tau-\varepsilon}^{-1} \right\| \le c\ (\alpha, \beta = 0,\dots,n;\ \tau > \varepsilon;\ \varepsilon > 0),$$

where c is the constant which does not depend on τ, and thus

$$\left\| (D^* - \mu I)^{-1} B_{\alpha\beta} v_\beta \right\|_\tau \le c \left\| v_\beta \right\|_{\tau-\varepsilon}$$

$$(\alpha, \beta = 0,\dots,n;\ \tau > \varepsilon;\ \varepsilon > 0),$$

c is constant.

Thus the property (vi) is true. The theorem 1 is proof.

If the polynomial $A(\xi) - \lambda$ has also null solutions (let be for example $\xi_0 = 0$ with $m_0 > 0$), theorem 1 can be refined, more exactly the following theorem is true.

Theorem 2. Let H be an operator of type (1), which acts in the space $L_2(R_+)$ and let be $q_{\alpha\beta}(x) = 0\ (x \in R_+)$ and $k_{\alpha\beta}(x, y) = 0\ (x, y \in R_+)$ for every $\alpha, \beta = 0,\dots,n$, with $\alpha + \beta < \eta$ (η an integer fixed number $0 < \eta < 2n$).

Let be that $A(\xi) - \lambda$ can be represented on the form (8), where we consider that $\xi_0 = 0$ and $m = \max\{m_0 - \eta, m_1, \dots, m_r\}$.

If

$$(1 + x)^\delta q_{\alpha\beta}(x) \in L_\infty(R_+)$$

$$(\delta > m;\ \alpha, \beta = 0,\dots,n;\ with\ \eta < \alpha + \beta < 2n)$$

and if $k_{\alpha\beta}(x, y) = 0$ for $x > y$, and the integral operator with kernels

$$(1 + x)^\delta k_{\alpha\beta}(x, y)$$

$$(\delta > m;\ \alpha, \beta = 0,\dots,n;\ with\ \eta < \alpha + \beta < 2n)$$

are bounded in $L_2(R_+)$, then λ is not an eigenvalue of the operator H.

Proof. The operator B can be represented of type $B = \sum_{\eta < \alpha + \beta < 2n} D^{*\alpha} B_{\alpha\beta} D^\beta$,

and the symbol of the operator A is

$$A(\xi) = \xi^{m_0} \prod_{j=1}^{r} (\xi - \xi_j) A_1(\xi).$$

We estimate the expressions of the form

$$D^{*\gamma} \prod_{j=1}^{r} D^{*-m_0} (D^* - \xi_j)^{-m_j} D^{*\alpha} (q_{\alpha\beta}(x) + K_{\alpha\beta}) v_\beta =$$

$$= D^{*\alpha+\gamma-m_0} \prod_{j=1}^{r} (D^* - \xi_j)^{-m_j} (q_{\alpha\beta}(x) + K_{\alpha\beta}) v_\beta,$$

where $\eta < \alpha + \beta < 2n;\ \gamma = 0,\dots,n;\ v_\beta \in L_2(R_+)$ and this can be represented like a linear combination of the form

$$(D^* - \mu I)^{-l} (q_{\alpha\beta}(x) + K_{\alpha\beta}) v_\beta$$

$$(l = 1,\dots,m;\ \eta < \alpha + \beta < 2n).$$

3 Applications

3.1. In a space $L_p(R_+)\ (1 \le p < \infty)$ consider the integro-differential operator

$$(Hu)(x) = -\frac{d^2u}{dx^2} + q_1(x)\frac{du}{dx} + q_0(x)u(x) + $$
$$+ \int_{R_+} k(x, y)u(y)dy\ \ (0 < x < \infty;\ u \in W_p^2(R_+)),$$

where $q_j\ (j = 0,1)$ are the measurable functions on the positive semi-axis R_+ and tend to zero when x tend to infinite, and the integral operator K with kernel $k(x, y)$ is supposed bounded in $L_2(R_+)$. The definition domain $D(H)$ of the operator H is considered the set of all u derivatives functions on positive semi-axis with derivative u' absolutely continuous on every bounded interval of the positive semi-axis and there is the derivative of the two order u'' there is on almost all semi-axis R_+, $u'' \in L_2(R_+)$ (in the sense of distributions) and $u(0)=0$.

In concordance with the theorem 1, we obtain the following result.

Corollary. Let be

$$(1+x)^\delta q_j(x) \in L_\infty(R_+)\,(j=0,1)$$

and let be that the kernel $k(x,y)$ of the integral operator K is such that $k(x,y)=0$ for $x>y$ (close on all semi-axis R_+) and the integral operator with kernel $(1+x)^\delta k(x,y)$ is bounded on the space $L_2(R_+)$.

If $\delta>1$, then the operator H has not eigenvalues on the positive semi-axis, and if $\delta>2$ then the point $\lambda=0$ also is not an eigenvalue of the operator H.

We mention that the operator H considered in the example 3.1 with

$$q_1(x)\equiv 0 \text{ and } k(x,y)=r(x)\cdot r(y),$$

where $r(x)=a\cdot x\cdot e^{-\alpha x}\ (\alpha, a \in R; \alpha>0)$, it is studied in the paper [15] in connection with the problems about the diffusion theory of neutrons in protons.

3.2. Let consider the operator

$$(Hu)(x) = -\frac{d^2u}{dx^2} + q_1(x)\frac{du}{dx} + \int_{R_+} k(x,y)\frac{du}{dy}dy \ \left(0<x<\infty; u \in W_p^2(R_+)\right).$$

H acts in the space $L_2(R_+)$. In concordance with the theorem 2 the following affirmation is true.

Corollary. If

$$(1+x)^\delta q_1(x) \in L_\infty(R_+)\,(\delta>1),$$

$k(x,y)=0$ for $x>y$, and if the integral operator with kernel $(1+x)^\delta k(x,y)\,(\delta>1)$, is bounded on $L_2(R_+)$, then the point spectrum of the operator H on the positive semi-axis (inclusive the point $\lambda=0$) is absent.

Similar results can be formulated for instance for the integro-differential operators which can be obtained as a result of the Schrödinger type operators (in general nonselfadjoints, see for instance [16]) perturbed with integral operators.

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