Trichotomous-noise-induced stochastic resonance for a fractional oscillator

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Abstract: Influences of the memory exponent and the flatness of a multiplicative noise on the long-time behavior of the output signal of a fractional oscillator with fluctuating eigenfrequency subjected to an external periodic force are considered. The colored fluctuations of the oscillator frequency are modeled as a three-level Markovian telegraph noise. The main purpose of this work is to demonstrate, based on exact expressions, that an interplay of colored noise and memory can generate a variety of cooperation effects, such as hypersensitive response of the output signal to noise amplitude at high values of noise flatness as well as friction-induced reentrant transitions between different resonance regimes of the oscillator. Particularly, a critical memory exponent has been found, which marks a dynamical transition in the behavior of the system.

Key–Words: Stochastic oscillator, fractional oscillator, stochastic resonance, trichotomous noise, hypersensitive response, viscoelastic friction.

1 Introduction

A popular generalization of the harmonic oscillator, called the fractional oscillator, consists in replacement of the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law type memory [1]–[4]. The main advantage of this equation is that it provides a physically transparent and mathematically tractable description of the stochastic dynamics in systems with slow relaxation processes and with anomalous slow diffusion (subdiffusion). Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [5, 6], viscoelastic media [7], and amorphous semiconductors [8]. Particularly, diffusion of mRNAs and ribosomes in the cytoplasm of living cells is anomalously slow [9], large proteins behave similarly [10]. Even intrinsic conformational dynamics of protein macromolecules can be subdiffusive [11, 12].

In most model systems described as a fractional oscillator, the effect of a fluctuating environment on dynamical equations is taken into account as an additive white noise or fractional noise. However, it is well recognized that there are some important systems, especially in the context of biological applications, where the influence of a fluctuating environment should be modeled as a multiplicative colored noise, which has a non-zero correlation time [13]–[16]. Although the behavior of the fractional oscillator with an additive noise has been investigated in detail, it seems that analysis of the potential consequences of an interplay between multiplicative noise and memory effects is still missing in literature. This is quite surprising, because the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g., for living cells, has been well recognized [10, 16].

Thus motivated, we consider a fractional oscillator with a power-law memory kernel. The influence of the fluctuating environment is modeled by a multiplicative three-level Markovian noise (trichotomous noise). Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noise [17, 18]. Furthermore, it is remarkable that for trichotomous noises the flatness parameter $\kappa$ can be anything from 1 to $\infty$, unlike the flatness for Gaussian colored noise, $\kappa = 3$, and symmetric dichotomous noise, $\kappa = 1$. This extra degree of freedom can prove useful in modeling actual fluctuations.

The main contribution of this paper is as follows. We provide exact formulas for the analytic treatment of the dependence of the mean oscillator displacement in the long-time limit, $t \to \infty$, on system parameters. On the basis of those exact expressions we will show that stochastic resonance (SR) is manifested in...
the dependence of the response of the noisy fractional oscillator upon the noise amplitude. To avoid misunderstandings, let us mention that we use the term SR in the wide sense, meaning nonmonotonic behavior of the output signal or some function of it, e.g., moments, in response to noise parameters [19]. Furthermore, we will show that at high values of noise flatness the output signal of the oscillator exhibits a hypersensitive response to noise amplitude. Moreover, we have found a critical memory exponent below which friction-induced reentrant transitions between different SR regimes of the oscillator appear.

The structure of the paper is as follows. In section 2 we present the basic model investigated. Exact formulas for the mean oscillator displacement are derived. In section 3 we analyze the behavior of the output response, and expose the main results of this paper. Section 4 contains some brief concluding remarks.

2 Model and the exact solution

As a model for an oscillatory system strongly coupled with a noisy environment, we consider a trichotomically perturbed oscillator with a power law type memory friction kernel

$$\ddot X + \gamma \frac{d^\alpha}{dt^\alpha} X + [\omega^2 + Z(t)]X = A_0 \sin(\Omega t),$$

where $\dot X \equiv dX/dt$, $X(t)$ is the oscillator displacement, $\gamma$ is a friction constant, and the fractional Caputo derivative with the memory exponent (fractional exponent) $0 < \alpha < 1$ is defined as in [20],

$$\frac{d^\alpha}{dt^\alpha} X := \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\dot X(t')}{(t - t')^\alpha} dt',$$

where $\Gamma(y)$ is the gamma function. Fluctuations of the eigenfrequency $\omega$ are expressed as a trichotomous process $Z(t)$ [17]. This is a random stationary Markovian process that consists of jumps between three values $a$, 0, and $-a$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$p_s(a) = p_s(-a) = q, p_s(0) = 1 - 2q,$$

with $0 < q \leq 1/2$. The mean value of $Z(t)$ and the correlation function are

$$\langle Z(t) \rangle = 0, \langle Z(t + \tau)Z(t) \rangle = 2qa^2 e^{-\nu \tau}.$$  

It can be seen that the switching rate $\nu$ is the reciprocal of the noise correlation time $\tau_c$, i.e., $\tau_c = 1/\nu$. The flatness parameter $\kappa$ of the noise $Z(t)$ proves to be a very simple expression of the probability $q$

$$\kappa := \frac{\langle Z^4(t) \rangle}{\langle Z^2(t) \rangle^2} = \frac{1}{2q},$$

As in this work we will restrict ourselves to the behavior of the first moment of the oscillator displacement ($X(t)$), all results are also applicable in models where an additive noise $\xi(t)$, which is statistically independent from $Z(t)$ and has a zero mean, is included in the right side of Eq. (1). For example, depending on the physical situation, the noise $\xi(t)$ can be regarded either as an internal noise, in which case its stationary correlation satisfies Kubo’s second fluctuation-dissipation theorem [21] expressed as

$$\langle \xi(t + \tau)\xi(t) \rangle = \frac{k_B T \gamma}{\Gamma(1 - \alpha) \tau^{-\alpha}}$$

(here $k_B$ is the Boltzmann constant and $T$ is the temperature of the heat bath), or as an external noise, in which case the driving noise $\xi(t)$ and the dissipation may have different origins and no fluctuation-dissipation relation holds.

To find the first moment of $X$, we use the well-known Shapiro-Loginov procedure [22], which for a trichotomous noise $Z(t)$ yields

$$\frac{d}{dt} \langle Z\Phi \rangle = \langle Z \frac{d}{dt} \Phi \rangle - \nu \langle Z\Phi \rangle,$$

where $\Phi$ is an arbitrary functional of the process $Z(t)$. From Eqs. (1) and (7), we thus obtain an exact linear system of six first-order integro-differential equations for six variables, $x_1 = \langle X \rangle$, $x_2 = \langle \dot X \rangle$, $x_3 = \langle XZ \rangle$, $x_4 = \langle X^2 \rangle$, $x_5 = \langle Z^2 X \rangle$, $x_6 = \langle Z^2 \rangle$:

$$\dot x_1 = x_2,$$

$$\dot x_2 = -\omega^2 x_1 - x_3 - \gamma \frac{d^\alpha}{dt^\alpha} x_1 + A_0 \sin(\Omega t),$$

$$\dot x_3 = -\nu x_3 + x_4,$$

$$\dot x_4 = -\nu x_4 - \omega^2 x_3 - x_5 - \gamma e^{-\nu t} \frac{d^\alpha}{dt^\alpha} (e^{\nu t} x_3),$$

$$\dot x_5 = -\nu x_5 + x_6 + 2qa^2 x_1,$$

$$\dot x_6 = 2qa^2 x_2 - \nu x_6 - 2qa^2 x_2 - a^2(1 - 2q)x_3,$$

$$-\omega^2 (x_5 - 2qa^2 x_1) - \gamma e^{-\nu t} \frac{d^\alpha}{dt^\alpha} \left[ e^{\nu t} (x_5 - 2qa^2 x_1) \right].$$

(8)
The solution of equations (8) can be formally represented in the form
\[ x_i(t) = \sum_{k=1}^{6} H_{ik}(t)x_k(0) + A_0 \int_{0}^{t} \left[ H_{k2}(t') + 2qa^2 H_{k6}(t') \right] \sin \left[ \Omega (t - t') \right] dt', \]
where the constants of integration \(x_k(0)\) are determined by the initial conditions. The relaxation functions \(H_{ik}(t)\) with the initial conditions \(H_{ik}(0) = \delta_{ik}\) can be obtained by means of the Laplace transformation technique. Particularly, we find that
\[ \hat{H}(s) := \hat{H}_{12}(s) + 2qa^2 \hat{H}_{16}(s) = \frac{1}{D(s)} \times \left\{ \left[ (s + \nu)^2 + \gamma(s + \nu)^{\alpha} + \omega^2 \right] - (1 - 2q)a^2 \right\}, \]
where
\[ D(s) = \left( 1 - 2q \right)a^2 \left[ \gamma(s + \nu)^{\alpha} - \gamma \alpha \nu + 2s + \nu \right] + \left[ (s + \nu)^2 + \gamma(s + \nu)^{\alpha} + \omega^2 \right] \left\{ \left( s^2 + \gamma \alpha \nu + \omega^2 \right) - a^2 \right\}, \]
and \(\hat{H}_{ik}(s)\) is the Laplace transform of \(H_{ik}(t)\), i.e.,
\[ \hat{H}_{ik}(s) = \int_{0}^{\infty} e^{-st} H_{ik}(t) dt. \]
One can check the stability of solution (9), which, according to the results of Ref. [23], means that the solutions \(x_i\) of the equation \(D(s) = 0\) cannot have roots with a positive real part. This requirement is met if the inequality
\[ a^2 < a_{cr} = \frac{\omega^2(\omega^2 + \nu^2 + \gamma \nu^{\alpha})}{(\omega^2 + 2q(\nu^2 + \gamma \nu^{\alpha}))} \]
holds. Henceforth in this work we shall assume that the condition (13) is fulfilled. Thus in the long-time limit, \(t \to \infty\), the memory about the initial conditions will vanish as
\[ \sum_{k=1}^{6} H_{ik}(t)x_k(0) = \frac{\gamma \hat{h}(0)x_1(0)}{\Gamma(1 - \alpha)} \frac{t^{\alpha - 1}}{t^{a}} + O \left( t^{-(1+\alpha)} \right) \]
and the average oscillator displacement \(\langle X \rangle_{as} \equiv \langle X \rangle_{t \to \infty}\) is given by
\[ \langle X \rangle_{as} = A_0 \int_{0}^{t} h(t - t') \sin(\Omega t') dt'. \]
From Eq. (15) it follows that the complex susceptibility \(\chi(\Omega)\) of the dynamical system (1) is given by
\[ \chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = \hat{h}(-i\Omega), \]
where \(\chi'(\Omega)\) and \(\chi''(\Omega)\) are the real and the imaginary parts of the susceptibility, respectively. Equation (15) can be written by means of the complex susceptibility as
\[ \langle X \rangle_{as} = A \sin(\Omega t + \phi) \]
with the output amplitude
\[ A = A_0 \cdot |\chi| \]
and the phase shift
\[ \phi = \arctan \left( -\frac{\chi''}{\chi'} \right) . \]
The exact formulas for \(A\) and \(\phi\), being complex and cumbersome for the general case, will not be presented here. To obtain some insight into the behavior of \(\langle X \rangle_{as}\) at various system parameter regimes we restrict ourselves to the adiabatic noise (i.e., \(\nu \to 0\)) At the long-correlation-time limit \(\tau_c \to \infty\), the output amplitude \(A\) is given by
\[ A^2 = \frac{A_0^2 \left[ (f_1 - (1 - 2q)a^2)^2 + f_2^2 \right]}{f_3 \left[ (f_1 - a^2)^2 + f_2^2 \right]}, \]
where
\[ f_1 = f_4^2 - \gamma^2 \Omega^{2\alpha} \sin^2 \left( \frac{\pi \alpha}{2} \right), \]
\[ f_2 = 2\gamma \Omega^\alpha f_4 \sin \left( \frac{\pi \alpha}{2} \right), \]
\[ f_3 = f_4^2 - \gamma^2 \Omega^{2\alpha} \sin^2 \left( \frac{\pi \alpha}{2} \right), \]
\[ f_4 = \omega^2 - \Omega^2 + \gamma \Omega^\alpha \cos \left( \frac{\pi \alpha}{2} \right) \].

3 Stochastic resonance

Our next task is to examine the dependence of the response \(A\) on the noise amplitude \(a\). In Fig. 1 we depict the behavior of \(A(a)\) for various values of the system parameters. As is shown in Fig. 1, curves (1) - (3) exhibit a resonance-like maximum at some values of \(a\), i.e., a typical SR phenomenon appears at increase of \(a\). The existence of such a SR effect depends strongly on other system parameters. From Eqs. (20) and (21) one can easily find the necessary and sufficient conditions for the emergence of SR due to noise amplitude variations. Namely, nonmonotonic behavior of \(A(a)\)
appears in the stability regions, $0 < a < a_{cr}$ (see Eq. (13)), for the parameter regime where the following inequalities hold:

$$2(1 - q)f_3^2\omega^4 > f_1 \left[f_3^2 + (1 - 2q)\omega^8\right] > 0.$$  (22)

In this case the response $A(a)$ reaches the maximum at

$$a_m^2 = \frac{f_3}{f_1(1 - 2q)} \left[(1 - q)f_3 - \sqrt{q^2f_3^2 + (1 - 2q)f_3^2}\right].$$  (23)

In Fig. 2 the conditions (22) are illustrated in the parameter space $(\gamma, \alpha)$ with two panels. The dark grey shaded domains in the figure correspond to those regions of the parameters $\gamma$ and $\alpha$, where SR versus $a$ is possible. Note that in the light grey regions the response $A(a)$ formally also exhibits a resonance-like maximum, but in those regions the first moment of the oscillator displacement $\langle X(t) \rangle$ is unstable, see Eq. (13). In the dark grey domain (the stability region) a stochastic resonance for $A$ vs $a$ occurs. The thin dashed line depicts the position of the critical memory exponent $\alpha_c = 1/2$. Panel (a): $\Omega = 0.6$; panel (b): $\Omega = 1.8$.

marks a sharp transition in the behavior of systems with fractional dynamics. At $\alpha_c$, one of the boundaries $\gamma(\alpha)$ between the resonance and no-resonance regions tends to infinity. The second finding is that depending on the driving frequency $\Omega$, two different cases can be discerned. (i) For $\Omega^2 < \omega^2$, resonance vs $a$ appears in the stability region for all values of $\gamma$ when $\alpha < \alpha_c$, but if $\alpha > \alpha_c$, there is an upper border $\gamma(\alpha)$ above which the resonance is absent (Fig. 2(a)). (ii) In the case of $\Omega^2 > \omega^2$, if $\alpha < \alpha_c$, the interesting peculiarity of the diagram is that there are two disconnected regions (the shaded areas in Fig. 2(b)) where the resonance can appear. Thus, in this case a variation of values of the friction parameter $\gamma$ induces reentrant transitions between different dynamical regimes.
in a region of hypersensitive response \[\text{[Eqs. (20) and (24)]}\). System parameters values: \(\gamma = 3 \cdot 10^{-4}, q = 5 \cdot 10^{-3}, \Omega = 0.99, \alpha = 0.1, \text{ and } A_0 = \omega = 1\). The value of \(A^2\) at the local maximum is \(A^2_{m} = 15857; A^2 = A^2/A^2_{m}\). of the oscillator. Namely, an increase of \(\gamma\) can induce transitions from a regime where SR vs \(a\) is possible to the regime where SR is absent, but SR appears again through a reentrant transition at higher values of \(\gamma\).

Next we consider, in brief, another interesting SR phenomenon – hypersensitive response to noise amplitude. A peculiarity of Fig. 3 is the rapid decrease of \(A^2\) from maximum to minimum as \(a\) increases. It is noteworthy that in the case of dichotomous noise such an effect is absent. The effect is very pronounced at low values of the damping constant \(\gamma\). To throw some light on the above-mentioned effect, we shall now briefly consider the behavior of the SR characteristic \(A^2\) in the parameter regime

\[
\gamma \Omega^\alpha << q \left| \omega^2 - \Omega^2 \right| < \omega^2, q << 1.  \tag{24}
\]

In this case, it follows from Eqs. (20) and (21) that \(A^2\) reaches the maximum

\[
A^2_{\text{max}} \approx A^2_0 \frac{q^2}{\gamma^2 \Omega^{2\alpha}} \tag{25}
\]

at

\[
a = a_{\text{max}} \approx \left| \omega^2 - \Omega^2 \right|,  \tag{26}
\]

and the minimum

\[
A^2_{\text{min}} \approx A^2_0 \frac{\gamma^2 \Omega^{2\alpha}}{q^2 (\omega^2 - \Omega^2)^4} \tag{27}
\]

at

\[
a = a_{\text{min}} \approx \frac{\left| \Omega^2 - \omega^2 \right|}{\sqrt{1 - 2q}}. \tag{28}
\]

For sufficiently strong inequalities (24), \(A^2_{\text{min}}\) tends to zero and \(A^2_{\text{max}}\) grows up to very large values. Thus in the case considered the response \(A\) is extremely sensitive to a small variation of \(a\): \(\Delta a = a_{\text{min}} - a_{\text{max}} \approx q^2 \left( \Omega^2 - \omega^2 \right)\).

Formulas (24) – (28) are exactly the same as can be derived from the results of [19] for an ordinary stochastic oscillator (without memory, \(\alpha = 1\)) if we replace the friction coefficient \(\gamma\) with the corresponding effective quantity \(\gamma_{\text{ef}} = \gamma \Omega^{\alpha-1}\). This suggests that the physical explanation of the effect of hypersensitive response to noise amplitude exposed in [19] for a stochastic oscillator without memory is also applicable in the case of a stochastic fractional oscillator.

4 Conclusions

In the present work, we have analysed the phenomenon of stochastic resonance within the context of a noisy, fractional oscillator with a fluctuating eigen-frequency driven by a sinusoidal forcing. The viscoelastic type friction kernel with memory is assumed as a power-law function of time and the eigen-frequency fluctuations are modeled as a colored three-level Markovian noise. The Shapiro-Loginov formula [22] with the Laplace transformation technique allow us to find an exact expression for the long-time behavior of the mean oscillator displacement.

As one of the main results we have established the effect of a very sensitive response of the mean oscillator displacement to small variations of the noise amplitude at high values of the noise flatness, i.e., the amplitude of the output signal displays a quick jump from a very high value to a low one as the noise amplitude increases but a little. It is important to note that such a phenomenon was previously reported for a stochastic oscillator without memory in Ref. [19]. As another main result we have found the existence of a band gap for the values of the friction coefficient \(\gamma\) between two regions of \((\gamma - \alpha)\) phase diagrams where SR vs the noise amplitude is possible at sufficiently small values of the memory exponent \(\alpha < 0.5\), and the corresponding friction-induced reentrant transitions between these different dynamical regimes of the oscillator.

We believe that the results obtained are of interest also in cell-biology, where issues of memory and multiplicative colored noise can be crucial [2, 10, 12, 16].

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