Vibrations of stepped cylindrical shells with cracks

JAAN LELLEP, LARISSA Roots
Institute of Mathematics, University of Tartu
2 Liivi str. Tartu
ESTONIA
jaan.lellep@ut.ee, a42119@ut.ee

Abstract: - Assuming that the deformations are axisymmetric the vibrations of stepped cylindrical shells are studied. Simple tool for the vibration analysis of shells accounting for the influence of cracks is presented. The changes of flexibility of the shell near cracks are prescribed by means of the compliance coupled with the stress intensity factor defined in the linear elastic fracture mechanics. The shell wall has an arbitrary number of steps and circular cracks of constant depth.

Key words: - crack, cylinder, vibration, elasticity, axisymmetric shell

1 Introduction

Plates and shells are often used in structures which are subjected to impact loading and vibrations. This involves the need for investigation of structures accounting for the influence of cracks on vibration of structural members like cylindrical shells.

During last decades a considerable attention has been paid to the investigation of vibration and stability of elastic beams with cracks.

Vibration problems are considered making use of the concept of additional compliance due to a crack developed by Dimarogonas (1996), also Chondros, Dimarogonas, Yao (1998, 2001). The case of inclined edge or internal cracks was studied by Nandwana and Maiti (1997).

The effect of cracks on the free vibration frequencies of uniform beams with arbitrary number of cracks was investigated by Lin, Chang, Wu (2002) by the use of the transfer matrix method.

In the present paper an attempt is made to extend the method of distributed springs to free axisymmetric vibrations of elastic circular cylindrical shells. The shells under consideration have piece wise constant thickness and circular cracks of constant depth are located at cross sections with steps of the thickness.

2 Formulation of the problem

Let us consider a circular cylindrical shell of length $l$ and radius $R$ (Fig.1). We shall confine our attention to axisymmetric free vibrations of the shell caused by an initial excitation.

![Fig.1: Stepped cylindrical shell](image-url)
3 Basic equations

Equilibrium conditions of a shell element have the form (see Reddy, 2007; Soedel, 2004; Ventsel and Krauthammer, 2001)

\[ \frac{\partial^2 M}{\partial x^2} - \frac{N}{R} + p - \rho h_j \frac{\partial^2 w}{\partial t^2} = 0 \]  

(1)

for \( x \in (a_j, a_{j+1}) \), \( j = 0, \ldots, n \).

Here \( M \) is the bending moment and \( N \) the circumferential membrane force and \( w \) – the transverse deflection (Soedel, 2004, Reddy, 2007). The generalized Hooke’s law with the equilibrium equation (1) yields the equation

\[ \frac{\partial^2 w}{\partial x^2} + \frac{12(l - v^2)}{R^2 h_j} w = - \frac{12\rho \cdot h_j (1 - v^2)}{E h_j} \frac{\partial^2 w}{\partial t^2} \]

(2)

which must be satisfied for \( x \in (a_j, a_{j+1}) \), \( j = 0, \ldots, n \).

Since \( N_i = 0 \) and \( \epsilon_1 = -\nu \epsilon \) one has in the present case

\[ N = E h_j \frac{w}{R} \]

for \( x \in (a_j, a_{j+1}) \).

4 Local flexibility of the shell

Let us consider the crack located at the cross section \( x = a_j \) and let the segments adjacent to the crack have thicknesses \( h_{j-1} \) and \( h_j \) respectively. According to the current approach it is assumed that the slope of deflection \( w \) is discontinuous at \( x = a_j \). Let the jump of the slope be \( \theta_j \). The quantity \( \theta_j \) can be treated as an additional angle caused by the crack at \( x = a_j \), whereas

\[ \theta_j = C_j M(a_j) \]

(3)

It is known that (Reddy, 2007; Anderson, 2005)

\[ \theta_j = \frac{\partial U_T}{\partial M(a_j)} \]

(4)

where \( U_T \) is the extra strain energy due to the crack. Making use of (3) and (4) one can easily recheck that

\[ C_j = \frac{\partial^2 U_T}{\partial M^2(a_j)} \]

(5)

Evidently, equalities (4), (5) coincide with those known in the linear elastic fracture mechanics, provided \( M \) is the generalized force and \( \theta_j \) the generalized displacement, respectively.

In the linear elastic fracture mechanics (see Anderson, 2005; Boberg, 1999) the stress intensity factor is defined as

\[ K = \sigma \sqrt{\pi c} \cdot F\left( \frac{c}{h} \right) \]

(6)

where \( F \) is a function to be defined experimentally. It can be shown, that the compliances \( C_j \) must be calculated as (see Dimarogonas, 1996; Lellep, Sakkov, 2006)

\[ C_j = \frac{72\pi}{E h_j^2} f(s_j) \]

(7)

where \( s_j = c/h_j \) and \( E = E \) for plane stress state and \( E = E(1-v^2) \) for plane deformation state. Thus one can state that

\[ C_j = \frac{72\pi}{E h_j^2} f(s_j) \]

(8)

Many authors have investigated the problem of determination of the stress intensity factor for various specimens (among others Brown and Srawley, 1966; Tada, Paris, Irwin, 2000).

In the present study we are resorting to the data of experiments conducted by Brown and Srawley which can be approximated as (see Tada, Paris, Irwin, 2000)

\[ F(s_j) = 1.93 - 3.07 s_j + 14.53 s_j^2 - 25.11 s_j^3 + 25.8 s_j^4 \]

(9)

Making use of (7) - (9) one obtains

\[ f(s_j) = 1.862 s_j^2 - 3.95 s_j^3 + 16.375 s_j^4 - 37.226 s_j^5 + 76.81 s_j^6 - 126.9 s_j^7 + 172.5 s_j^8 - 143.97 s_j^9 + 66.56 s_j^{10} \]

(10)

The function (10) is employed also in papers by Dimarogonas (1996), Chondros, Dimarogonas and Yao (2001), Kukla (2009), also by Lellep and Sakkov (2006). According to the concept of massless rotating spring one can equalize \( K = 1/C_j \) and thus

\[ K_j = \frac{E h_j^2}{72\pi \cdot f(s_j)} \]

(11)

where \( f(s_j) \) is defined by (10) for \( s = s_j \).

5 Solution of governing equations

Let us assume that for \( x \in (a_j, a_{j+1}) \)

\[ w(x,t) = X_j(x)T(t) \]

(12)
where \( X_p(x) \) and \( T=T(t) \). Differentiating (12) with respect to \( x \) and \( t \) and substituting in (2) leads to the equation

\[
X_j''''(x)T(t) + \frac{12(1-v^2)}{R^2h_j^2} X_j(x)T(t) = -\rho \cdot \frac{12(1-v^2)}{Eh_j^2} X_j(t)T(t),
\]

(13)

where the notation

\[
X_j''''(x) = \frac{d^4X_j}{dx^4},
\]

\[
T(t) = \frac{d^2T}{dt^2},
\]

is used. These equations must be satisfied for \( x \in (a_j, a_{j+1}) \) for \( j=0, ..., n \).

Separating variables in (13) one easily obtains

\[
\ddot{T} + \omega^2 T = 0
\]

(14)

and

\[
X_j'''' - r_j^4 X_j = 0
\]

(15)

for \( x \in (a_j, a_{j+1}), j=0, ..., n \). Here

\[
r_j^4 = \omega^4 \cdot \frac{12\rho(1-v^2)}{Eh_j^2} \cdot \frac{12(1-v^2)}{R^2h_j^2},
\]

(16)

where \( \omega \) stands for the frequency of free vibrations of the shell.

Equation (14) has in the case of the initial condition \( T(0)=0 \) the solution

\[
T = \alpha \sin(\omega t),
\]

(17)

\( \alpha \) being arbitrary constant.

The general solution of the linear fourth order equation (15) can be presented as

\[
X_j(x) = A_j \sin(r_j x) + B_j \cos(r_j x) + C_j \sinh(r_j x) + D_j \cosh(r_j x),
\]

(18)

for \( x \in (a_j, a_{j+1}), j=0, ..., n \).

If, for instance, edges of the shell are simply supported the boundary requirements give conditions

\[
X_j(0) = 0, \quad X_j''(0) = 0
\]

(19)

The rotating spring model entails discontinuities of the slope \( w'(x,t) \) as it was discussed above. Evidently, the bending moment \( M(x,t) \) is continuous, if \( h^3w'''(x,t) \) is continuous. Similarly we can conclude that the shear force is continuous if the quantity \( h^2w''(x,t) \) is continuous when passing the cross sections \( x=a_j \).

Thus one obtains the intermediate conditions at

\[
x = a_j, \quad j=0, ..., n
\]

\[
X_j(a_j + 0) - X_j(a_j - 0) = 0,
\]

\[
X_j'(a_j + 0) - X_j'(a_j - 0) + p_j X_j''(a_j - 0) = 0,
\]

\[
h_j^3 X_j''(a_j + 0) - h_j^3 X_j''(a_j - 0) = 0,
\]

\[
h_j^3 X_j''(a_j + 0) - h_j^3 X_j''(a_j - 0) = 0
\]

(21)

where the notation

\[
p_j = \frac{Eh_j^3}{12(1-v^2)K_j}
\]

(22)

is introduced.

According to (16) it is reasonable to introduce a real number \( k \) so that

\[
r_j = \frac{k}{\sqrt{h_j}}
\]

(23)

for each \( j=0, ..., n \).

Let us proceed to the determination of constants \( A_p, B_p, C_p, D_p \) in (18). If, for instance, the left end of the shell is simply supported then

\[
B_0 = D_0 = 0
\]

(24)

However, in the case of the clamped right end one has

\[
A_n \sin r_n l + B_n \cos r_n l = 0,
\]

\[
C_n \sinh r_n l + D_n \cosh r_n l = 0
\]

(25)

Intermediate conditions have the form
In order to solve the system (26) with appropriate boundary requirements imposed on \(X_0(0)\) and \(X_0(l)\) let us introduce the notation

\[
\begin{bmatrix}
A_j \\
B_j \\
C_j \\
D_j
\end{bmatrix}
\]

The system of equations (26) can be rewritten in the form

\[
M_{j+1} \tilde{Y}_{j+1} = N_j \tilde{Y}_j
\]

for \(j=0,\ldots,n\), where

\[
N_j = \begin{bmatrix}
\sin r_j a_j & \cosh r_j a_j \\
-r_j(\cosh r_j a_j + p_j r_j \sin r_j a_j) & \sin r_j a_j \\
-h_j^3 r_j^2 \sin r_j a_j & -h_j^3 r_j^2 \sin r_j a_j \\
-h_j^3 r_j^2 \cos r_j a_j & h_j^3 r_j^2 \cos r_j a_j
\end{bmatrix}
\]

\[
M_{j+1} = \begin{bmatrix}
\sin r_{j+1} a_j & \cosh r_{j+1} a_j \\
-r_{j+1}(\cosh r_{j+1} a_j + p_{j+1} r_{j+1} \sin r_{j+1} a_j) & \sin r_{j+1} a_j \\
-h_{j+1}^3 r_{j+1}^2 \sin r_{j+1} a_j & -h_{j+1}^3 r_{j+1}^2 \sin r_{j+1} a_j \\
-h_{j+1}^3 r_{j+1}^2 \cos r_{j+1} a_j & h_{j+1}^3 r_{j+1}^2 \cos r_{j+1} a_j
\end{bmatrix}
\]

and

\[
\tilde{Y}_0 = \begin{bmatrix}
A_0 \\
0 \\
C_0 \\
0
\end{bmatrix}
\]

\[
\tilde{Y}_j = \mathbf{P} \tilde{Y}_{j-1}
\]

where

\[
\mathbf{P} = S_n S_{n-1} \ldots S_1
\]

and

\[
S_j = N_j^{-1} M_{j+1}
\]

Equations (28)-(34) can be converted into a system of linear homogeneous algebraic equations with respect to unknown constants. Since the determinant of this system vanishes one obtains the characteristic equation

\[
(p_{j+1} \sin r_{j+1} l + p_{j+1} \cos r_{j+1} l)(p_{j+1} \sin r_{j+1} l + p_{j+1} \cos r_{j+1} l) - (p_{j+1} \sin r_{j+1} l + p_{j+1} \cos r_{j+1} l)(p_{j+1} \sin r_{j+1} l + p_{j+1} \cos r_{j+1} l) = 0
\]

which can be easily solved with the aid of existing codes with respect to the characteristic number \(k\) related to \(r_j\) by relations (23).

### 6 Numerical results

Calculations have been carried out for shells with one and two steps and with simply supported right and left ends. Results of calculations are presented in...
Figs. 2-6. Figs. 2-5 correspond to shells with thicknesses \( h_0 \) and \( h_1 \) having a single step of the thickness. It is assumed herein that the material of the shell is a homogeneous elastic material.

In calculations cylindrical shells of length \( l=1,2m \) clamped at the left hand are considered. It has been taken \( h_0=0,009m \).

The influence of the non-dimensional crack depth \( c/h \) on the characteristic number \( k \) is depicted in Fig.2. Here and henceforth we take \( h=min(h_0, h_1) \). Let us introduce the notations \( \gamma=h_1/h_0, \beta=a_1/l \).

In Fig.2 the sensitivity of the number \( k \) (and thus the eigen frequency \( \omega \)) with respect to crack length \( c \) and the location of the crack \( a_1=\beta l \) is shown. Here \( h_0= h_1 \), e.g. the shell is of constant thickness. It can be seen from Fig.2 that the bigger is the crack length the smaller is the number \( k \).

For instance, the lowest eigenfrequencies (the smallest number \( k \)) correspond to the case \( \beta=0,6 \).

In Fig.3 the number \( k \) is shown for stepped shells with \( h_1=0,3 h_0 \). Here different curves correspond to different locations of the step, e.g. to different values of \( \beta=a_1/l \). It can be seen from Fig.3 that the number \( k \) decreases when the crack length \( c \) increases in the case of a fixed value of \( \beta \). It is interesting to note that in the range of small cracks the number \( k \) is almost unsensitive with respect to small changes of the crack length.

However, in the range of larger cracks when \( c>0,6 h_1 \), this sensitivity is more obvious. Similar matters can be observed in Figs.4,5 , as well.

In Figs.4 the dependence between the characteristic number \( k \) and the crack length \( c \) is shown for stepped shells with step coordinates \( a_1=0,2l \) respectively. It can be seen from Fig. 4 that when the step is near to the left end (here \( \beta=0,2 \)) then the number \( k \) weakly depends on the crack length \( c \). If, however, \( a_1 \) is larger then this dependence is more remarkable, especially in the case of large cracks.

Similarly to previous results one can see from Fig. 4 that in the case of small cracks the number \( k \) is almost unsensitive to the crack length.

In Fig. 5 the results corresponding to shells with two cracks and with simply supported right hand end and clamped at the left hand end are presented.
Fig. 5: Two stepped cylindrical shell, the case $\gamma_1=0.2$; $\gamma_2=0.5$.

Curves 1, 2, 3, 4 in Fig. 5 are associated with the crack locations at $a_2=0.3l$; $a_2=0.4l$; $a_2=0.7l$; $a_2=0.9l$, respectively, whereas $a_1=0.1l$.

Thus the numerical results show that for fixed values of geometrical parameters the number $k$ decreases when crack length $c$ increases.

7 Concluding remarks

Calculations showed that the cracks have essential influence on vibration characteristics. It was established that if the crack location is fixed then the maximal value of the characteristic number $k$ is achieved if the crack length is equal to zero. On the other hand, for fixed crack length and variable ratio of thicknesses maximum of the number $k$ is achieved when $h_0=h_1$ thus for the shell of constant thickness. When ratio of thicknesses is fixed whereas $h_1<h_0$ then the minimum of $k$ is achieved when the step and crack location tend to the free end.

Acknowledgement

The support from Estonian Research Foundation (grant ETF 7461) is acknowledged.

References: