About root-clustering in sophisticated regions

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Abstract: This paper presents a new criteria for matrix root-clustering in sophisticated, multiply-connected and non-convex regions of the complex plane. A concept of the root-clustering region to be the complement of the root-clustering region to \( \mathbb{C} \) and the concepts of the forbidden subregions and the expanded root-clustering regions are introduced to formulate the main results of the paper. The set of the modified three-parametrical Cassini regions are offered to use as the subregions covering the forbidden region. In these terms the generalizations of the classical Gutman theorem and the Jury-Ahn theorem onto the intersection of the systems of nonlinear algebraic inequalities are obtained. An application to the problem of root-clustering in outside of a system of the forbidden frequency bands is shown for illustration.

Key–Words: Matrix root clustering, Lyapunov inequality, Forbidden subregions, expanded root-clustering regions, Modified Cassini regions

1 Introduction

The properties of control systems strongly depend on the location of eigenvalues of a system matrix on the complex plane. In this connection the problem of root-clustering acquires interest and importance for control systems analysis and has received attention. This paper is devoted to a problem of matrix root-clustering in the sophisticated regions given as an intersection of several algebraic regions of the fourth order and smaller order. Such regions can have one or several boundaries. The boundaries may be smooth or piecewise smooth, convex or nonconvex; the boundaries may be formed by segments of algebraic curves or by curves enveloping a continual set of curves. The region may be multiply- or simply-connected, non-convex or convex.

2 Root-clustering in outside of the closed forbidden region \( \bar{S} \).

In the literature a problem of matrix root-clustering is usually considered as a problem of a membership of all roots to a specified open region \( D \) of the complex plane \( \mathbb{C} \). Let us consider another treatment of this problem. Let the region \( S \) be the complement to \( \mathbb{C} \) of \( D \), i.e., \( \bar{S} = \mathbb{C} \setminus D \); then the region \( S \) is called the forbidden region.

Let us consider 2 cases.

Case 1 Suppose the region \( \bar{S} \) is defined as below:

\[
\bar{S} = \bigcup \bar{S}_\alpha \quad \forall \quad \alpha = \bar{1}, \nu,
\]

where \( \{ \bar{S}_\alpha, \quad \forall \alpha = \bar{1}, \nu \} \) is a final set of mutually intersecting or nonintersecting subregions; the envelope of this set forms the border of \( \bar{S} \). Then by definition put

\[
\{ D_\alpha : \quad D_\alpha = \mathbb{C} \setminus \bar{S}_\alpha \supseteq D \} \quad \forall \quad \alpha = \bar{1}, \nu,
\]

where the regions \( D_\alpha \) are called the expanded root-clustering regions. Note that \( D_\alpha \supseteq D \); this follows from \( S_\alpha \subseteq \bar{S} \Rightarrow D_\alpha = \mathbb{C} \setminus S_\alpha \supseteq \mathbb{C} \setminus \bar{S} = D \).

Case 2 Let \( \{ \bar{\Omega}(\mu), \mu \in [0, \tau] \} \) be a continual set of the subregions covering the region \( \bar{S} \)

\[
\bar{S} = \bigcup \bar{\Omega}(\mu) \quad \forall \quad \mu \in [0, \tau]
\]

Then by definition put

\[
\{ D(\mu) : \quad D(\mu) = \mathbb{C} \setminus \bar{\Omega}(\mu) \} \quad \forall \quad \mu \in [0, \tau],
\]

where \( D(\mu) \supseteq D \quad \forall \mu \in [0, \tau] \) is the continual parametric set of the expanded root-clustering regions.

Theorem 1 Let the forbidden region \( \bar{S} \) for the matrix spectrum \( s(A) \) be a union of the regions (1); then the root-clustering region \( D = \mathbb{C} \setminus \bar{S} \) satisfies the equality

\[
D = D^*, \quad D^* = \bigcap D_\alpha = \bigcap (\mathbb{C} \setminus \bar{S}_\alpha), \quad \alpha = \bar{1}, \nu
\]

i.e., \( D \) is the intersection of the set of the expanded root-clustering regions (2).
Theorem 2 Let the forbidden region \( S \) be a union of the parametric continual set (3); then the root-clustering region \( D \) satisfies the equality
\[
D = D^* = \cap D(\mu), \quad D(\mu) = \mathbb{C} \setminus \Omega(\mu). \tag{6}
\]

2.1 Generalization of root-clustering theorems.

The methods first developed by Gutman, Jury and colleagues [1] and then continued in a number of works, some of them are [2], [3], [4] [5], enlarge Lyapunov theory on the regions defined by algebraic inequalities. This paper offers the further generalization of these results on a case of a final set of algebraic inequalities. By \( A' \) denote the transpose of a matrix \( A \in \mathbb{R}^{n \times n} \). Suppose the forbidden subregion \( \bar{S}_\alpha \) in the union (1) for all \( \alpha = 1, \nu \) satisfy the inequalities
\[
\bar{S}_\alpha = \{ s \in \mathbb{C} : f_\alpha(s) = |\psi_\alpha(s)|^2 - |\varphi_\alpha(s)|^2 \geq 0 \}, \tag{7}
\]
where \( f_\alpha(s) \neq \text{const} \), and \( \psi_\alpha(s), \varphi_\alpha(s) \) are some polynomials of \( s \). Then the expanded regions \( D_\alpha = \mathbb{C} \setminus \bar{S}_\alpha \) satisfy the inequalities
\[
D_\alpha = \{ s \in \mathbb{C} : f_\alpha(s) = |\psi_\alpha(s)|^2 - |\varphi_\alpha(s)|^2 < 0 \}. \tag{8}
\]

Theorem 3 Let the root-clustering region \( D \) be defined as (5), (8). Then a necessary and sufficient condition for the matrix spectrum \( s(A) \) to be clustered in \( D \) is that there exist \( \nu \) symmetric positively determined matrices \( X_\alpha \) satisfying the matrix inequalities
\[
\bar{\psi}_\alpha(A')X_\alpha\psi_\alpha(A) - \bar{\varphi}_\alpha(A')X_\alpha\varphi_\alpha(A) < 0, \quad X_\alpha > 0. \tag{9}
\]

Proof: Actually, any region \( D_\alpha \) from set (8) satisfies the Jury-Ahn theorem; therefore conditions (9) are necessary and sufficient conditions for root-clustering in any \( D_\alpha \) from the set (8). In light of Theorem 1 the region \( D \) is the intersection of the final set of the expanded regions (8). From here follows, that a necessary and sufficient condition for root-clustering in \( D \) is the intersection of conditions (9) for all \( \alpha = 1, \nu \). This completes the proof of Theorem 3 \( \Box \).

Further, let’s consider the forbidden region \( \bar{S} \) covered by a continual set of intersecting and non-intersecting parametric subregions \( \bar{S}(\mu) \). In other words, let the subregion \( \bar{S}(\mu) \) completely sweeps the region \( \bar{S} \) as parameter \( \mu \) ranges over \([0, \nu]\), i.e.,
\[
\bar{S}(\mu) = \{ s \in \mathbb{C} : f_\alpha(s) = |\psi(s, \mu)|^2 - |\varphi(s, \mu)|^2 \geq 0 \}. \tag{10}
\]

Theorem 4 Assume that the open region \( D \) is a complement to \( C \) of the forbidden region \( \bar{S} = \cup \bar{S}(\mu) \forall \mu \in [0, \tau] \), where the continual set of the forbidden subregions \( \bar{S}(\mu) \) is determined by inequality (10). The spectrum \( s(A) \) lies in the region \( D = \mathbb{C} \setminus \bar{S} \) if and only if there exists a symmetric positively determined depending on parameter \( \mu \) matrix \( X(\mu) > 0 \) such that at all values of the parameter \( \mu \) it satisfies to a matrix inequality
\[
\bar{\psi}(A', \mu)X(\mu)\psi(A, \mu) - \bar{\varphi}(A', \mu)X(\mu)\varphi(A, \mu) < 0. \tag{11}
\]

These obtained theorems can be easily generalized on a case of complex functions \( \psi \) and \( \phi \) and on a case of a complex matrix \( A \).

Theorem 5 Let the forbidden region is given as an association of subregions (1)
\[
\bar{S}_\alpha = \{ s \in \mathbb{C} : \sum_{0 \leq k,l \leq m} c_{kl}^\alpha s^k \bar{s}^l \geq 0 \}, \quad c_{kl}^\alpha = c_{lk}^\alpha \in \mathbb{R}. \tag{12}
\]
It corresponds to the root-clustering region
\[
D = \cap D_\alpha \quad \forall \alpha = 1, \nu, \tag{13}
\]
where \( D_\alpha = \{ s \in \mathbb{C} : \sum_{k,l} c_{kl}^\alpha s^k \bar{s}^l < 0 \} \). The spectrum \( s(A) \) is clustered in \( D \), if and only if there exist a set of symmetric matrices \( X_\alpha, \forall \alpha = 1, \nu \) satisfying
\[
\sum_{k,l} c_{kl}^\alpha A^k X_\alpha(A') \bar{s}^l < 0, \quad X_\alpha > 0, \quad \forall \alpha = 1, \nu \tag{14}
\]

3 Modified Cassini regions.

As it is known, the equation of Cassini ovals includes two parameters \((a, c)\) determining the shape of the ovals. Let us introduce the modified Cassini ovals. First we replace \( c^2 \) by \( a \) and \( a^2 \) by \((-c)\) in the equation of the Cassini ovals; here the negative mark results in an orientation of the ovals along an imaginary axis at \( c > 0 \) or orientation along a real axis at \( c < 0 \). Secondly we introduce a third parameter \( \mu \geq 0 \) providing the parallel shift of the ovals along the real axis to the left-hand side. Thus we obtain the modified tree-parametric Cassini ovals defined by the equation
\[
\Gamma(a, c, \mu) = \{ s = x + jy : ((x + \mu)^2 + y^2)^2 + 2c((x + \mu)^2 - y^2) + c^2 - a^2 = 0 \}, \tag{15}
\]
where \( a > 0, \mu \geq 0, c \in [-a/2, 0] \cup [a, \infty) \).
The restriction $\mu \geq 0$ means that we consider the shift of the ovals only to the left-hand side, and the restriction $c \in [-a/2, 0] \cup [a, \infty)$ means that among different forms of Cassini lines we shall consider only ovals.

Equation (15) in terms of a complex variable $s$ has the form

$$\Gamma = \{ s \in \mathbb{C} : ((s + \mu)^2 + c)((s + \mu)^2 + c) - a^2 = 0 \},$$

where $a > 0$, $\mu \geq 0$, $c \in [-a/2, 0] \cup [a, \infty)$. Note that at $c = 0$ we have the unit circle, at $c < 0$ we obtain the oval with semi-axis $(\sqrt{a - c})$ and $(\sqrt{a + c})$, extended along the real axis. At $c = -a/2$ the curvature of the oval in the top and in the bottom points are equal to zero. At $c = a$ we have the lemniscate. Finally, at any values $c > a$ we have the pair of ovals symmetrically located at both sides of the real axis having the height of $(\sqrt{c + a} - \sqrt{c - a})$ and having the maximal width of $(a/\sqrt{c})$.

Suppose the closed region $H$ satisfies the inequality

$$\bar{H}(a, c, \mu) = \{ s \in \mathbb{C} : |(s + \mu)^2 + c^2 - a^2| \leq 0 \},$$

where $a > 0$, $\mu \geq 0$, $c \in [-a/2, 0] \cup [a, \infty)$. Then the region $\bar{H}(a, c, \mu)$ is called the modified three-parametric Cassini oval region. It is clear that curve (16) is the boundary of $\bar{H}$.

Further, the complement of $H$ to $\mathbb{C}$ is called the external oval region and denoted by $G$. The region $G$ satisfies the inequality

$$G(a, c, \mu) = \{ s \in \mathbb{C} : f(s) = a^2 - |(s + \mu)^2 + c|^2 < 0 \},$$

where $a > 0$, $\mu \geq 0$, $c \in [-a/2, 0] \cup [a, \infty)$.

### 3.1 Conformal mapping of Cassini regions.

The important property of the modified Cassini ovals making them suitable for application in root-clustering problems rests on the fact that the function of a complex variable

$$z = a[((s + \mu)^2 + c)^{-1}, \quad c > a > 0, \quad \mu \geq 0, \quad (19)$$

conformally maps the closed Cassini oval region $H(a, c, \mu)$ onto the complement of the closed unit central disk $\{ z : |z| \geq 1 \}$ [6]. The reader will easily check that substituting (19) for (17), (18). Note also that function (19) maps the complementary region $G(a, c, \mu) = \mathbb{C} \setminus \bar{H}(a, c, \mu)$ into the open unit disk $\{ z : |z| < 1 \}$. The bilinear function

$$w = (z + 1)(z - 1)^{-1}, \quad (20)$$

conformally maps the unit central disk $\{ z : |z| < 1 \}$ and the exterior disk $\{ z : |z| \geq 1 \}$ respectively into the left half-plane and onto the right half-plane. Substituting (20) for (19), we get

$$w(a, c, \mu) = (a + c + (s + \mu)^2)(a - c - (s + \mu)^2)^{-1}, \quad (21)$$

This function conformally maps the open region $G(a, c, \mu)$ and the closed region $\bar{H}(a, c, \mu)$ respectively into the open left half-plane and onto the right half-plane.

### 3.2 Root-clustering theorems for oval subregions.

Suppose the forbidden region $\bar{S}$ is covered by the final set of regions (17), i.e.,

$$\bar{S} = \cup \bar{H}(a, c, \mu), \quad \forall \alpha = \frac{1}{\nu}, \quad (22)$$

where

$$\bar{H}(a, c, \mu) = \{ s \in \mathbb{C} : |(s + \mu)^2 + c| - a^2 \leq 0 \}, \quad (23)$$

Then the set of external oval regions $G(a, c, \mu)$ is defined as follows

$$G = \{ s : f(s) = a^2 - |(s + \mu)^2 + c|^2 < 0 \}. \quad (24)$$

Let $\psi_\alpha(s)$ and $\phi_\alpha(s)$ in inequality (7) be defined as

$$\psi_\alpha(s) = a_\alpha, \quad \phi_\alpha(s) = (s + \mu_\alpha)^2 + c.$$ 

Then by Theorem 3, we get the following statement.

**Theorem 6** All roots of a real matrix $A$ lie in the region $D = \mathbb{C} \setminus S$, where $S$ is the forbidden region (22), (23), if and only if there exist a set of positively definite symmetric matrices $\{ X_\alpha, \forall \alpha = \frac{1}{\nu} \}$ satisfying

$$a_\alpha^2X_\alpha - ((A^T + \mu_\alpha E)^2 + c_\alpha E)X_\alpha((A + \mu_\alpha E)^2 + c_\alpha E) < 0. \quad (25)$$

Let the parameters $a$ and $c$ be functions of the third parameter $\mu$, and the oval region completely sweeps the forbidden region as $\mu$ ranges over $[0, \tau]$, i.e.,

$$\bar{S} = \cup \bar{H}(\mu), \quad (26)$$

$$\bar{H}(\mu) = \{ s : a^2(\mu) - |(s + \mu)^2 + c(\mu)|^2 \geq 0 \}.$$ 

Then using Theorem 4, we get the following statement.
Theorem 7 All roots of a real matrix $A$ are clustered in $D = \mathbb{C} \setminus \hat{S}$, where $\hat{S}$ is the forbidden region (26), if and only if for all $\mu \in [0, \tau]$ there exists a positively definite symmetric parameter-depending matrix $X(\mu) > 0$ satisfying the following parametric matrix inequality

$$a^2 X(\mu) - ((A' + \mu E)^2 + + cE) X(\mu) ((A + \mu E)^2 + cE) < 0 \quad (27)$$

The following functionally transformed matrices correspond to functions (19) and (21):

$$Z_{\alpha} = a_{\alpha}((A + \mu_\alpha E)^2 + c_{\alpha} E)^{-1} \forall \alpha \in [1, \nu] \quad (28)$$

$$W_\alpha = ((a_{\alpha} + c_{\alpha}) E + (A + \mu_\alpha E)^2)((a_{\alpha} - - c_{\alpha}) E - (A + \mu_\alpha E)^2)^{-1} \quad (29)$$

Using the property of matrix eigenvalues we get the following theorem.

Theorem 8 All roots of a real matrix $A$ are clustered in $D = \mathbb{C} \setminus \hat{S}$, where $\hat{S}$ is the forbidden region (22), (23), if and only if all eigenvalues of all functionally transformed matrices (28) are clustered in the unit central disk, or if all eigenvalues of all matrices (29) are clustered in the left half-plane.

Theorem 8 leads to the Hurwitz test of matrices (29) or Lyapunov inequality for them; or this theorem leads to the conditions of Schur-Cohn for matrices (28).

4 Example

This example is adopted from a band-filtration problem and illustrates the developed method. Let a band filter has three suppression frequency bands: low, high and intermediate all having different width. In terms of the developed method these bands form the forbidden region. Now we will obtain the algebraic conditions for matrix root-clustering in outside of this forbidden region. First we approximate the forbidden bands by a set of modified three-parametric Cassini oval regions (22),(23). The low frequency band we approximate by four simply-connected oval regions with the values of the parameters $a = 1, c = -0.5, \mu = [0, 2, 4, 6]$. The intermediate and high bands we approximate by multiply-connected oval regions with the values of the parameters $\{a = 10, c = 14.5, \mu = [0, 2, 4, 6]\}$ and $\{a = 6.5, = 42.5, \mu = [0, 0.7, ..., 6.3]\}$. As a whole we approximate the forbidden region by seventeen simply-connected and conjugate oval regions. Secondly in the light of Theorem 7 all roots of matrix $A$ are clustered in outside of this forbidden region, if and only if there exist seventeen positively-definite symmetric matrices $\{X_{\alpha}, \forall \alpha = 1,17\}$ satisfying (25) for all given above values of the parameters $a_{\alpha}, c_{\alpha}, \mu_{\alpha}$.

5 Conclusion

In this paper the new theorems for matrix root-clustering in sophisticated and multiply-connected regions with piecewise-smooth boundary have been established. We introduce a new approach to the problem of root-clustering based on the application of a new class of regions of the forth order. The example illustrates an application of the developed method to the problem of band filtration.

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References:


