Qualitative analysis of ODEs describing cavitation erosion

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Abstract. Analytical models of curves describing the cavitation erosion were developed using ordinary differential equations (ODEs). Many existing models of cavitation erosion give special attention to the volume loss (or weight loss) and the volume loss rate curve (or weight loss rate curve), usually described as solutions of ODEs. Some methods used until now, are focused on finding the ODEs whose solutions depending on real parameters, describe how much better the characteristic curves of cavitation erosion. The parameters involved, are determined by fitting the erosion curve to the experimental data, with numerical methods. In this paper using geometrical methods in order to obtain qualitative information about the behavior of the solutions of ODEs, it is proposed a qualitative study of the ordinary differential equations whose solutions can describe the characteristic curves of cavitation erosion.

Key-words: Phase space, Phase portrait, Equilibrium, Cavitation erosion, Hamiltonian.

1. Introduction.
Cavitation erosion is a progressive loss of material from a solid, due to the impact action of the collapsing bubbles or cavities into liquid near the material surface. Cavitation erosion can be formed when cavity implosions are violent enough and they take place near enough to the solid material. So, mechanical degradation of a solid material caused by cavitation is called cavitation erosion.
The cavitation erosion has been evaluated quantitatively based on cumulative mass loss, cumulative volume loss and cumulative mean depth of penetration. The volume loss is often preferred for comparison of materials with great differences between the specific masses.
An important effort of researchers in the field of cavitation is focused on improving the techniques and analytical models of prediction of cavitation erosion.
J. Noschevich and K. Steler [6], [9], [10], proposed models of erosion kinetics referring directly to the erosion curve pattern. The volume loss curve is obtained by solving the ordinary differential equation with constant coefficients:
\[ \frac{d^2 v}{dt^2} + 2\alpha \frac{dv}{dt} + \beta^2 v = I \] (1)
(with \( V(t) \) the volume loss at time \( t \), \( v(t) = \frac{d(V(t))}{dt} \) the volume loss rate, \( \alpha \) and \( \beta \) - coefficients determining material properties, \( I \)- cavitation intensity parameter), or the differential equation:
\[ \frac{1}{P} \frac{d^2 v}{dt^2} + \frac{2\alpha}{P} \frac{dv}{dt} + \beta^2 v = I = \gamma P \] (2)
(with \( P \) the power of flux energy delivered by the cavitation cloud to the eroded material, \( \gamma \) - the coefficient defined by equation (2)).

2. Qualitative study of autonomous ODEs describing cavitation erosion.
A typical ODE of second order, with unknown function \( x(t) \) describing cavitation erosion, can be replaced with a system of first order :
\[ \begin{align*}
\dot{x} &= f(x, y, \alpha, \beta) \\
\dot{y} &= g(x, y, \alpha, \beta)
\end{align*} \] (3)
where are \( \alpha, \beta, \ldots \) real constant.
The plane with coordinates \( x \) and \( y \) is the phase space or phase plane of the given ODE.
The points of the phase space are called phase points. The right-hand side of (3) determines a vector field on the phase plane, called the phase velocity vector field.
A solution of (3) is a motion \( \varphi : \mathbb{R} \to \mathbb{R}^2 \) of a phase point in the phase plane, such that the velocity of the moving point at each moment of time is equal to the phase velocity vector at the location of the phase point at that moment.
The image of \( \varphi \) is called the phase curve. Thus the phase curve is given by the parametric equations \( x = \varphi(t), y = \dot{\varphi}(t) \).
When a phase curve consist of only one point, such a point is called *equilibrium position*. The vector of phase velocity at an equilibrium position is zero. So, the equilibrium point are obtained by solving the system.

\[
\begin{align*}
    f(x, y, \alpha, \beta...) &= 0 \\
    g(x, y, \alpha, \beta...) &= 0
\end{align*}
\] (4)

Depending on the eigenvalues of the Jacobian matrix

\[
J(\alpha, \beta,...) = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\]

at the equilibrium point, we can find properties of the system and possible bifurcation points.

Using the symplectic structure of phase space can determine whether the system (3) is conservative. Recall that the phase space has a symplectic form \(\omega = dx \wedge dy\). A vector field \(X\) on phase space is called Hamiltonian if there is a smooth function \(H: \mathbb{R}^2 \rightarrow \mathbb{R}\), representing the total energy,

\[
H(x, y) = T(v) + V(x)
\] (5)

\((T\) denote the kinetic energy of the system and \(V\) is the potential energy of the system), such that

\[
i_v \omega = dH
\]

\((i_v \omega\) denotes the interior product or contraction of the vector–field \(X\) and the 2–form \(\omega\). \(X\) is locally Hamiltonian if \(i_v \omega\) is closed.

The Hamiltonian vector–field \(X_H\), condition by \(i_{X_H} \omega = dH\), is defined via symplectic matrix

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{as}
\]

\[
X_H = J \Delta H = (\partial_H - \partial_X H)
\] (6)

where \(\Delta\) is the gradient operator.

The law of conservation of energy allows one to find the phase curves easily taking into account that on each phase curve the value of Hamiltonian is constant. Therefore, each phase curve lies entirely in one energy level set \(H(x, y) = C\).

To note that if the potential energy is positive, there is a one-parameter group of diffeomorphisms of the phase plane- the phase flow of vector field \(X\).  

### 3. Example. Qualitative study of Noschevich and Steler ODE describing cavitation erosion

Let \(\frac{d^2v}{dt^2} + 2\alpha \frac{dv}{dt} + \beta^2 v = I\), be the ODE proposed by Noschevich and K. Steler, with \(V(t)\) the volume loss at time \(t\), \(v(t) = \frac{d(V(t))}{dt}\) the volume loss rate, \(\alpha\) and \(\beta\) -coefficients determining material properties.

Putting in formula (1), \(v = x\) and \(\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}\) the ODE (1), can be written:

\[
\ddot{x} + 2\alpha \dot{x} + \beta^2 x = I
\] (7)

The ODE (1) is equivalent with autonomous system:

\[
\begin{align*}
    \dot{x} &= y \\
    \dot{y} &= I - 2\alpha y - \beta^2 x
\end{align*}
\] (8)

The equilibrium point is given by the system

\[
\begin{align*}
    y &= 0 \\
    I - 2\alpha y - \beta^2 x &= 0
\end{align*}
\] (9)

So the system (8) have an equilibrium point

\[
(\frac{I}{\beta^2}, 0)
\] (10)

The Jacobian matrix of (*) is:

\[
J(\alpha, \beta) = \begin{bmatrix} 0 & 1 \\ -\beta^2 & -2\alpha \end{bmatrix}
\] (11)

Their eigenvalues of \(J(\alpha, \beta)\) are:

\[
\lambda_1 = -\alpha + \sqrt{\alpha^2 - \beta^2}, \quad \lambda_2 = -\alpha - \sqrt{\alpha^2 - \beta^2}
\]

and the associated eigenvectors are respectively:

\[
\begin{align*}
    &\begin{bmatrix} 1 \\ -\alpha + \sqrt{\alpha^2 - \beta^2} \end{bmatrix} \\
    &\begin{bmatrix} 1 \\ -\alpha - \sqrt{\alpha^2 - \beta^2} \end{bmatrix}
\end{align*}
\]

Therefore the general solution of (4) is:

\[
\begin{align*}
    x(t) &= C_1 e^{-\alpha + \sqrt{\alpha^2 - \beta^2}} \left( \begin{array}{c} 1 \\ -\alpha + \sqrt{\alpha^2 - \beta^2} \end{array} \right) + \\
    y(t) &= C_2 e^{-\alpha - \sqrt{\alpha^2 - \beta^2}} \left( \begin{array}{c} 1 \\ -\alpha - \sqrt{\alpha^2 - \beta^2} \end{array} \right)
\end{align*}
\]

**Remark.** \(x = \frac{I}{\beta^2}\), is a particular solution of ODE (7), so the general solution of this equation is:

\[
\begin{align*}
    x(t) &= C_1 e^{-\alpha + \sqrt{\alpha^2 - \beta^2}} + C_2 e^{-\alpha - \sqrt{\alpha^2 - \beta^2}}
\end{align*}
\]
Remark. The plane with coordinates \(x\) and \(y\) is the phase plane of Equation (7). The points of this phase plane are phase points. The right-hand side of (8) determines the phase velocity vector field. A solution of (8) is a motion \(\varphi : \mathbb{R}^2 \to \mathbb{R}\) of a phase point in the phase plane. The image of \(\varphi\) is the phase curve. Thus the phase curve is given by the parametric equations \(x = \varphi(t), y = \dot{\varphi}(t)\).

If a phase curve could consist of only one point, such point is equilibrium position. The vector of phase velocity at an equilibrium position is zero.

Writing the (8) system in the form

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial y} = y \\
\dot{y} &= -\frac{\partial H}{\partial x} = I - 2\alpha y - \beta^2 x \\
\end{align*}
\]

we have

\[
\frac{dy}{dx} = \frac{I - 2\alpha y - \beta^2 x}{y} \quad (12)
\]

The solutions of (12) are the phase curves.

The system (8) is Hamiltonian if there is a differentiable function \(H : \mathbb{R}^2 \to \mathbb{R}\) such that

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial y} = y \\
\dot{y} &= -\frac{\partial H}{\partial x} = I - 2\alpha y - \beta^2 x \\
\end{align*}
\]

In our case we have

\[
\begin{align*}
\frac{\partial H}{\partial y} &= \beta^2 x^2 - Ix + k \\
\frac{\partial H}{\partial x} &= I - 2\alpha y - \beta^2 x \\
\end{align*}
\]

The first equation of (14) gives us

\[
H = \frac{1}{2} y^2 + f(x) \quad (15)
\]

Replaced (15) in the second equation of (14) we have:

\[-f'(x) = I - 2\alpha y - \beta^2 x \]

So, there is a Hamiltonian \(H\) if and only if \(\alpha = 0\) \((f'(x) = \beta^2 x - I)\) and

\[
f(x) = \frac{1}{2} \beta^2 x^2 - Ix + k, \quad (17)
\]

\(k\) a real constant. Finally:

\[
H = \frac{1}{2} y^2 + \frac{1}{2} \beta^2 x^2 - Ix + k \quad (18)
\]

Note that \(H\) can be interpreted as total energy. Putting

\[
T = \frac{1}{2} \dot{x}^2 = \frac{1}{2} y^2 \quad (19)
\]

(the kinetic energy), and

\[
U(x) = -\int_{x_0} f'(s)ds \quad (20)
\]

(the potential energy), we have that \(H = T + U\).

Proposition. (The law of conservation of energy). The Hamiltonian \(H\) is independent of time \(t\).

Proof. Differentiating the Hamiltonian (18) with respect to time we have:

\[
\frac{d}{dt} H = \frac{d}{dt} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \beta^2 x^2 - Ix + k \right)
\]

\[
= \ddot{x} \dot{x} + \beta^2 x \dot{x} - I \dot{x} = \dot{x}(\ddot{x} + \beta^2 x - I) = 0,
\]

because \(\ddot{x} + \beta^2 x = I\).

Remark. The law of conservation of energy allows us to find the phase curves taking into account that on each phase curve the value of Hamiltonian is constant and each phase curve lies entirely in one energy level set \(H(x, y) = C\).

In our case, the phase curve is given by

\[
C = \beta^2 x^2 - Ix + k = 0
\]

The Distance Formula.

Let \((\frac{1}{\beta^2}, 0)\) be equilibrium point of the system of differential equation (8) and \(\varphi(t) = (x(t), y(t))\) be a solution point. The distance between \((\frac{1}{\beta^2}, 0)\) and

\[
\varphi(t) = (x(t), y(t)) \neq (\frac{1}{\beta^2}, 0),
\]

is given by

\[
\rho\left(\varphi(t), (\frac{1}{\beta^2}, 0)\right) = \sqrt{[x(t) - \frac{1}{\beta^2}]^2 + [y(t)]^2} \quad (21)
\]

and its rate of change is given by:

\[
\frac{d}{dt}\rho\left(\varphi(t), (\frac{1}{\beta^2}, 0)\right) =
\]

\[
= \frac{1}{2} \frac{2[x(t) - \frac{1}{\beta^2}]\dot{x}(t) + 2y(t)\dot{y}(t)}{\sqrt{[x(t) - \frac{1}{\beta^2}]^2 + [y(t)]^2}}
\]
\[
\begin{align*}
\frac{1}{2} & \left( 2\left(x(t) - \frac{I}{\beta^2}\right)y(t) + 2y(t)(1 - \beta^2 x(t)) \right) \\
& = \sqrt{\left(x(t) - \frac{I}{\beta^2}\right)^2 + \left(y(t)\right)^2}
\end{align*}
\]

\[
y(t)\left(x(t) - \frac{I}{\beta^2} + I - \beta^2 x(t)\right) \\
= \sqrt{\left(x(t) - \frac{I}{\beta^2}\right)^2 + \left(y(t)\right)^2}
\]

\[
y(t)(1 - \beta^2)\left(x(t) - \frac{I}{\beta^2}\right) \\
= \sqrt{\left(x(t) - \frac{I}{\beta^2}\right)^2 + \left(y(t)\right)^2}
\]

(22)

Since \(\sqrt{\left(x(t) - \frac{I}{\beta^2}\right)^2 + \left(y(t)\right)^2} > 0\), putting

\[
\Delta(t) = y(t)(1 - \beta^2)\left(x(t) - \frac{I}{\beta^2}\right),
\]

the distance \(\rho\left(\varphi(t),\left(\frac{I}{\beta^2},0\right)\right)\), between \(\left(\frac{I}{\beta^2},0\right)\)

and \(\varphi(t) = (x(t),y(t))\), is:
- increasing if \(\Delta(t) > 0\);
- decreasing if \(\Delta(t) < 0\), that is, the trajectory spiral toward equilibrium.

If \(\Delta(t) = 0\), the distance \(\rho\left(\varphi(t),\left(\frac{I}{\beta^2},0\right)\right)\) is constant, and in this case the phase curves are circles, equilibrium point is a center.

4. Conclusion.

The notion of a dynamical system includes a set of its possible states (state space) and a law of the evolution of the state in time. The study of ODEs, using geometrical methods allow to obtain qualitative information about the behavior of the solutions of ODEs. In the case of cavitation erosion, the qualitative changes that occur in nature of solution of ODEs modeling this phenomena, when the parameters passes through a critical point, are very important.

We tried to propose explicit procedures for application of general mathematical theories to particular research problems.

The study of phase curves, bifurcations diagrams, etc., for different models and their comparison with experimental data, lead us to choose the optimal theoretical model in given situations.

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