Power Series Solutions of Boundary Layer Problem for Non-Newtonian Fluid Flow Driven by Power Law Shear

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Abstract: - In this paper we consider boundary layer similarity flow of a non-Newtonian power law fluid past an impermeable flat plate, driven by a power law velocity profile \( U = B y^\sigma \). We show that the momentum equation has power series solutions, valid for all the allowed range of the parameter \( \sigma \).

Key-Words: - Similarity solution, power series solution, boundary value problem, non-Newtonian fluid flow

1 Introduction
The problems of heat and mass transfer in two-dimensional boundary layers on continuous stretching surfaces, moving in an otherwise quiescent fluid medium, have attracted considerable attention during the last few decades. There are numerous applications in industrial manufacturing processes, such as rolling, wire drawing, glass-fiber and paper production, drawing of plastic films, metal and polymer extrusion and metal spinning.

The case of the laminar boundary layer to an exterior velocity profile of the form \( U = B y^\sigma \) was investigated by Weidman et al. [17] for a large range of value of the power law parameter \( \sigma \) and an analytical solution of the momentum equation in terms of Airy function was proposed for the case \( \sigma = -1/2 \). Then, Magyari et al. [15] found analytical solution for the same problem with permeable wall.

Recently, the study of non-Newtonian fluid flows has considerable interests, this is primarily because of the numerous applications in several engineering fields. One particular non-Newtonian model which has been widely studied is the Ostwald-deWaele power-law model, which relies the shear stress to the strain rate by the expression

\[
\tau_{xy} = \mu_0 \left( \frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial u}{\partial y},
\]

where \( \mu_0 \) is a positive constant, and \( n > 0 \) is called the power-law index. The case \( n < 1 \) is referred to pseudo-plastic or shear-thinning fluid, the case \( n > 1 \) is known as dilatant or shear-thickening fluid. The Newtonian fluid is a special case where the power-law index \( n = 1 \).

For Newtonian fluid with \( \sigma = 0 \) the problem of laminar boundary layer problem is described by the famous Blasius equation [3].

Our interest in this work has been motivated by the work of Cossali [9], who have considered the similarity flow over an impermeable flat plate driven by a power law velocity profile for Newtonian fluid, for which power series solutions were found for all the allowed range of the parameter \( \alpha \).

The objective of the present paper is to show the existence of power series solutions for the momentum boundary layer equation under the general case of the power-law velocity profile, thus extending the classical Blasius result for the no-shear case and non-Newtonian fluid flow when \( 0 < n < 2 \).

2 Derivation of the Problem
Consider a steady two-dimensional laminar flow of an incompressible fluid of density \( \rho \), past a semi-infinite flat plate. Let \((x, y)\) be the Cartesian coordinates of any point in the flow domain, where \( x \)-axis is along the plate and \( y \)-axis is normal to it. The continuity and momentum equations can be simplified, within the boundary-layer approximation, into the following equations [2]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}
\]

and

\[
u \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y}, \tag{2}
\]
where \( u \) and \( v \) represent the components of the fluid velocity in the direction of increasing \( x \) and \( y \), \( \tau_{xy} \) denotes the shear stress. Equations (1) and (2) are accompanied by the boundary conditions

\[
(3) \quad u(x,0) = 0, \quad v(x,0) = 0 \quad \text{and} \quad \lim_{y \to \infty} u(x,y) = U \quad \text{as} \quad y \to \infty,
\]

where \( U = By^\sigma \) as \( y \to \infty \). In term of the stream function \( \psi \), which satisfied \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \), equations (1), (2) can be reduced to the equation

\[
(4) \quad \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \mu_c \frac{\partial}{\partial y} \left[ \frac{\partial \psi}{\partial y} \right]^{n-1} \frac{\partial^2 \psi}{\partial y^2}
\]

where \( \mu_c = \frac{\mu B}{\rho} \), with the boundary conditions

\[
(5) \quad \frac{\partial \psi}{\partial y}(x,0) = 0, \quad \frac{\partial \psi}{\partial x}(x,0) = 0,
\]

and

\[
(6) \quad \lim_{y \to \infty} \frac{\partial \psi}{\partial y}(x,0) = U.
\]

To look for similarity solutions we define the stream function \( \psi \) and similarity variable \( \eta \) as

\[
(7) \quad \psi = b x^{-\alpha} f(\eta), \quad \eta = a x^{-\beta} y,
\]

where \( a, b, \alpha \) and \( \beta \) are constants to be determined, and \( f(\eta) \) denotes the dimensionless stream function. Using (4) and (7) we find that the profile function \( f \) satisfies

\[
(8) \quad \mu_c a^{2n+1} b^{2n} x^{-(\alpha+2\beta)n-\beta} \left( f^{n-1} \right)' - \alpha a^{2} b^{2} x^{-2(\alpha+\beta)-1} f \left( f'' \right) + (\alpha + \beta) a^{2} b^{2} x^{-2(\alpha+\beta)-1} f'^2 = 0
\]

Equation (8) is an ordinary differential equation if and only if \((2-n)\alpha + (1-2n)\beta = 1, \quad \alpha + \beta = M\); the scaling relation, i.e.,

\[
\alpha = \frac{M(2n-1)-1}{n+1}, \quad \beta = \frac{M(2-n)+1}{n+1},
\]

and the parameters \( a \) and \( b \) satisfy \( \mu_c a^{2n+1} b^{n-2} = 1 \), moreover

\[
M = -\frac{\sigma}{(2-n)\sigma + (n+1)}.
\]

So, the function \( f \) satisfies the following boundary value problem

\[
(9) \quad (f^{n-1})' - \alpha f f'' + M f'^2 = 0
\]

and

\[
(10) \quad f(0) = 0, \quad f'(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = A \eta^\sigma,
\]

where the prime denotes the differentiation with respect to the similarity variable \( \eta \), and \( A = B \left( a^{1+\sigma} b \right), \quad \sigma + 1 = -\alpha / \beta \). With the choice \( a = 1 \) we get that \( b = \mu_c^{1/(2-n)}, \quad A = B \mu_c^{-1/(2-n)} \), and the non-dimensional velocity components are obtained by \( f \) as follows:

\[
(11) \quad u(x, y) = \mu_c^{1/(2-n)} x^{-M} f'(\eta), \quad v(x, y) = x^{-(\alpha+1)} [\alpha f(\eta) + \beta \eta f'(\eta)].
\]

For the Newtonian case equation (11) reads

\[
(12) \quad \left( f^{n-1} \right)' + \frac{1}{2} f f'' = 0.
\]

If \( \sigma = 0 \) then (10) is reduced to

\[
(13) \quad f(0) = 0, \quad f'(0) = 0, \quad \lim_{\eta \to \infty} f'(\eta) = A.
\]

The boundary value problem (11) with (12) is called Blasius problem.

### 3 Power Series Solution

The Blasius function is defined as the unique solution to the boundary value problem (11)-(12). Blasius [3] derived power series expansion which begins
where \( \gamma \) is the curvature of the function at the origin. A closed form for the coefficients is not known. However, the coefficients can be computed for

\[
f(\eta) = \eta^2 \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k A_k \eta^{k+1} (3k+2)! \eta^{3k},
\]

and \( A_0 = A_1 = 1 \). Here \( \gamma \) is obtained numerically by Howarth [12] as \( \gamma \approx 0.332057 \). Abbasbandy [1] proposed an Adomian’s decomposition method to the Blasius’s problem and obtained \( \gamma = 0.333329 \) with a 0.383\% relative error, and Tajvidi et al. [16] used a modified rational Legendre method, to show that \( \gamma = 0.33209 \) with a 0.009\% relative error. By the fourth-order Runge–Kutta method \( \gamma \) is determined \( \gamma \approx 0.33205733621519630 \) (see [7]). A fully analytical solution of the Blasius problem has been found by Liao [14] using the homotopy analysis method.

Our object is to determine an approximate local solution \( f(\eta) \) of the boundary value problem (9)-(10). We use the shooting method and replace the condition at infinity by one at \( \eta = 0 \). Therefore, (10) is converted into an initial value problem with initial conditions

\[
f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \gamma.
\]

Let us suppose that \( 0 < n < 2 \), and \( f'' \) is positive in the neighborhood of zero. We will consider (9) as a system of certain differential equations, namely, the special Briot-Bouquet differential equations. For this type of differential equations we refer to the book by E. Hille [11], E. L. Ince [13]. In order to establish the existence of a power series representation of \( f(\eta) \) about \( \eta = 0 \) we apply the following theorem:

**Briot-Bouquet Theorem** [8]: Let us assume that for the system of equations

\[
\frac{\varepsilon d^2\xi}{d\xi^2} = u_1(\xi, z_1(\xi), z_2(\xi)),
\]

\[
\frac{d\xi}{d\xi} = u_2(\xi, z_1(\xi), z_2(\xi)),
\]

where functions \( u_1 \) and \( u_2 \) are holomorphic functions of \( \xi, z_1(\xi), \) and \( z_2(\xi) \) near the origin, moreover \( u_1(0,0,0) = u_2(0,0,0) = 0 \), then a holomorphic solution of (14) satisfying the initial conditions \( z_1(0) = 0 \), \( z_2(0) = 0 \) exists if none of the eigenvalues of the matrix

\[
\begin{bmatrix}
\frac{\partial u_1}{\partial z_1}(0,0,0) & \frac{\partial u_1}{\partial z_2}(0,0,0) \\
\frac{\partial u_2}{\partial z_1}(0,0,0) & \frac{\partial u_2}{\partial z_2}(0,0,0)
\end{bmatrix}
\]

is a positive integer. Briot-Bouquet theorem ensures the existence of formal solutions \( z_1 = \sum_{k=0}^{\infty} a_k \xi^k \) and \( z_2 = \sum_{k=0}^{\infty} b_k \xi^k \) for system (14), and also the convergence of formal solutions. This theorem has been successfully applied to the determination of local analytic solutions of different nonlinear initial value problems [4-6].

Let us consider the initial value problem (9)-(13), and take its solution in the form

\[
f(\eta) = \eta^2 Q(\eta^\delta), \quad \eta \in (0, \eta_c),
\]

where function \( Q \in C^2(0, \eta_c) \) for some positive value \( \eta_c \). Substituting \( f(\eta) = \eta^2 Q(\eta^\delta) \) into (9) one can get

\[
\delta^3 \eta^{3\delta-1} Q'' + 3 \delta^2 (\delta + 1) \eta^{2\delta-1} Q'' + \delta(\delta + 1)(\delta + 2) \eta^{\delta-1} Q' + M\left(2\eta Q + \delta \eta^{\delta+1} Q' \right)^2 - \frac{\alpha}{n} \eta^2 Q \left[2Q + \delta Q' + 3\eta \delta Q'' + \delta^2 \eta^{2\delta} Q''\right]^{3-n} = 0.
\]

Let us introduce the new variable \( \xi \) such as \( \xi = \eta^\delta \) and function \( Q \) as follows

\[
Q(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + z(\xi),
\]
where \( a_0, a_1, a_2 \) are real constants, and 
\[ z \in C^2(0, \eta^\delta), \quad z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 0. \]
Then \( Q \) fulfills the following properties \( Q(0) = a_0, \quad Q'(0) = a_1, \quad Q''(0) = 2a_2, \quad Q'''(0) = z''(0). \) From the initial condition \( f''(0) = 2Q(0) = \gamma \) and we have
\[ a_0 = \frac{\gamma}{2}. \quad (17) \]
We restate the third order differential equation in (18) as a system of three equations
\[
\begin{align*}
 u_1(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_2, \\
 u_2(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_3, \\
 u_3(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_3',
\end{align*}
\]
with choosing
\[
\begin{align*}
 z_1(\xi) &= z(\xi), \\
 z_2(\xi) &= z'(\xi), \\
 z_3(\xi) &= z''(\xi)
\end{align*}
\]
and
\[
\begin{align*}
 u_1(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_2, \\
 u_2(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_3, \\
 u_3(\xi, z_1(\xi), z_2(\xi), z_3(\xi)) &= \xi z_3',
\end{align*}
\]
where
\[ K(\xi) = 2a_0 + (2 + \delta(\delta + 3))a_1\xi + 2(\delta(\delta + 3)a_1 + (\delta^2 + 1)a_2)\xi^2 + 2z_1(\xi) + \delta(\delta + 3)z_2(\xi) + \delta^2\xi^2z_3(\xi). \]
We apply the Briot-Bouquet theorem for the system of three equations (18)-(20). In order to satisfy the conditions
\[ u_1(0, 0, 0, 0) = u_2(0, 0, 0, 0) = u_3(0, 0, 0, 0) = 0 \]
in the corresponding Briot-Boquet theorem the following connection yields
\[ \frac{3}{\delta} - 2 = -1, \]
therefore \( \delta = 3 \), and for the coefficients of \( \xi^{-1} \) and \( \xi^{0} \):
\[
\frac{\alpha}{27n} a_0(2a_0)^{2-n} - \frac{4M}{27} a_0 - \frac{20}{9} a_1 = 0, \quad (21)
\]
\[
\frac{\alpha}{27n} a_1(2a_0)^{2-n}(21 - 10n) - \frac{20M}{27} a_1 a_0 - \frac{112}{9} a_2 = 0, \quad (22)
\]
respectively. Applying (17) for (21) and (22)
\[ a_1 = \frac{1}{8!}\left(\frac{\alpha}{n} \gamma^{3-n} - 2M\gamma^2\right), \quad (23) \]
\[ a_2 = \frac{1}{8!}\left(\frac{\alpha(21 - 10n)}{n} \gamma^{2-n} - 10M\gamma\left(\frac{\alpha}{n} \gamma^{3-n} - 2M\gamma^2\right)\right) \quad (24) \]
Therefore the eigenvalues of matrix
\[
\begin{bmatrix}
\partial u_1 / \partial z_1 & \partial u_1 / \partial z_2 & \partial u_1 / \partial z_3 \\
\partial u_2 / \partial z_1 & \partial u_2 / \partial z_2 & \partial u_2 / \partial z_3 \\
\partial u_3 / \partial z_1 & \partial u_3 / \partial z_2 & \partial u_3 / \partial z_3 
\end{bmatrix}
\]
at \( (0, 0, 0) \) are 0. Since all three eigenvalues are non-positive, applying Briot-Bouquet theorem we obtain the existence of unique analytic solutions \( z_1, z_2 \) and \( z_3 \) near zero. Thus there exists a formal solution.
where the first three coefficients are known (see (17), (23), (24)).

For the determination of coefficients \( a_k, \ k > 2 \), we shall use the J.C.P. Miller formula [12]:

\[
\left[ \sum_{k=0}^{L} c_k x^k \right]^{p+1} = \sum_{k=0}^{(p+1)L} d_k (p) x^k,
\]

where \( d_0 (p) = 1 \) for \( c_0 = 1 \), and

\[
d_k (p) = \frac{1}{k} \sum_{j=0}^{k-1} [(p+1)(k-j) - j] d_j (p) c_{k-j}, \quad (k \geq 1).
\]

From (25)

\[
\left[ f''(\eta) \right]^{2-n} = \sum_{k=0}^{\infty} (3a_k + 2)(3a_k + 1) \eta^{3k}
\]

\[
= \sum_{k=0}^{\infty} A_k \eta^{3k},
\]

where coefficients \( A_k \) can be expressed in terms of \( a_k \) (\( k = 0, 1, \ldots \)). Substituting them into equation (9) we get

\[
\sum_{k=0}^{\infty} (3a_k + 5)(3a_k + 4)(3a_k + 3) \eta^{3k}
\]

\[
- \frac{1}{n} \alpha \sum_{k=0}^{\infty} a_k \eta^{3k} \sum_{k=0}^{\infty} A_k \eta^{3k}
\]

\[
+ M \left[ \eta \sum_{k=0}^{\infty} (3a_k + 2) \eta^{3k} \right]^{2} = 0.
\]

Applying recursion formula (26) for the determination of \( A_k \) and the comparison of the proper coefficients in (27) one can have the necessary values of \( a_k \) for some given values of \( n, M, \alpha \).

We note that the coefficients obtained this method for \( n = 1, \ \sigma \neq 0 \) are the same as the coefficients of the power series approximation given by G.E. Cossali [9]. Moreover, if \( n \neq 1, \ \sigma = 0 \) the coefficients \( a_k \) are fully consistent with the result obtained by [6]. If \( n = 1, \ \sigma = 0 \) the coefficients coincide with the Blasius results [3].

4 Conclusion

In this paper we have generalized the power series formulation of the similarity solution of the Newtonian flow over an impermeable flat plate driven by a power law velocity profile obtained by G.E. Cossali [9] to non-Newtonian fluid flow with Ostwald de Waele power-law nonlinearity when for the power-law index the condition \( 0 < n < 2 \) holds. The coefficients of the more general problem coincide with the coefficients of the problems related to special values of the parameters.

References:


