Improvement of the series solution of boundary layer problems in fluid flow using iterated Shanks transformation

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Abstract—The Blasius and Falkner–Skan equations arise in the study of laminar boundary layers exhibiting similarity. In this paper, the mathematical models for steady boundary layer flow past a horizontal flat plate and a semi-infinite wedge are considered. The nonlinear partial differential equations consisting of two independent variables are solved in the power series form. The radii of convergence of the solutions of the Blasius and Falkner-Skan equations are presented using the Domb-Sykes plot. The solution of the Falkner-Skan equation is based on the improvement of the perturbation series, which diverges beyond a certain radius, by means of the iterated Shanks transformation. The effectiveness of the method is illustrated by applying it successfully to various instances of the Falkner–Skan equation.

Keywords—Boundary layer, Domb-Sykes plot, Falkner-Skan problem, Shanks transformation.

I. INTRODUCTION

BOUNDARY layer over a flat plate is among the simplest and the earliest of the boundary layer flow case which has been developed. It was first established by Blasius [1], and he has found a series solution of the Blasius equation. However, some of the coefficients reported by Blasius in his 1907 Göttingen dissertation were wrong, and later he gave new values but one of them still wrong. Decades after, Weyl [2] established the complete Blasius series and presented the relation of the coefficients of the series. He also pointed out that the series has a finite radius of convergence in some range of values.

The Falkner-Skan problem, on the other hand, corresponds to the boundary layer flow past a semi-infinite wedge. It was first studied by V. W. Falkner and S. W. Skan in 1931. Rosenhead [3] and Weyl [4] present mathematical treatments of this problem focusing on existence and uniqueness results. Later, it was solved numerically by Hartree [5]. Cebeci and Keller [6] then applied the Newton method to solve the Falkner-Skan equation. Other numerical treatments included those developed by Smith [7] and Na [8]. These previous approaches have mainly used shooting and invariant imbedding. It is also important to note that the methods of Cebeci and Keller [6] have experienced convergence difficulties, which were overcome by moving towards more complicated methods. Recent methods presented by Asaithambi [9-11] have improved the performance of the previous methods greatly by reducing the amount of computational effort while obtaining the same results as the previous authors. One of the remarkable methods was used by Aziz and Na [12], that is, by using the Shanks transformation to improve the perturbation series of the Falkner Skan solution. The approach was motivated from Van Dyke [13], who discovered earlier the reliable of it.

In this paper, we will seek the solution of Blasius and Falkner-Skan equations, in the form of formal power series. Then the application of the Domb-Sykes plots will examine the radii of convergence of the series solutions. The main part of this paper describes the solution of the Falkner-Skan equation using the perturbation series which is improved further by Shanks transformation. The velocity profiles of the fluid flow can be obtained satisfactorily by only takes about five terms of the perturbation series.

II. PROBLEM FORMULATION

A. Boundary Layer Problem

The two-dimensional boundary layer flow of a viscous and incompressible fluid past an immersed body is considered. This problem is modelled in a rectangular Cartesian coordinate $(x, y)$ where $x$ is the coordinate measured along the surface body while $y$ is the coordinate measured along the normal direction to the surface body. The dimensionless boundary layer equations governing the steady flow are

\[
\begin{align*}
  \frac{\partial u}{\partial x} + \frac{v}{\partial y} &= u, \quad \frac{d u}{dx} + \frac{\partial^2 u}{\partial y^2} \\
  \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0
\end{align*}
\]

where $u$ and $v$ are the velocity component in the $x$ and $y$ direction respectively, and $u_e$ is the velocity of fluid in the inviscid region. Equations (1) and (2) are subject to boundary conditions
By introducing the stream function \( \psi \) defined as
\[
\psi = \frac{\partial u}{\partial y} \, , \quad v = \frac{-\partial u}{\partial x}
\]
it satisfies the continuity (2) identically. Hence, the boundary layer equations can be represented just by a single equation, i.e.
\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = u_x \frac{du_x}{dx} + \frac{\partial^3 \psi}{\partial y^3}
\]

### III. The Similarity Transformation

Equation (3) can be reduced to an ordinary differential equation using the similarity transformation
\[
\psi = u_0(x)g(x)f(\eta)
\]
where
\[
\eta = \frac{y}{g(x)}
\]
is the similarity variable. This transformation is applied considering on flow past a horizontal flat plate and a semi-infinite wedge, which leads to the Blasius equation and the Falkner-Skan equation respectively.

#### A. Blasius equation

Blasius [1] first suggested that \( \psi = (2x)^{1/2} f(\eta) \) in (4) where \( u_0(x) = 1 \) and \( g(x) = (2x)^{1/2} \), and the similarity variable is \( \eta = y/(2x)^{1/2} \). These result the Blasius problem which can be written as
\[
f^{m} + ff^{n} = 0
\]
subject to
\[
f(0) = f'(0) = 0 \quad , \quad f'(\infty) = 1
\]
Blasius [1] then obtained the series solution for the boundary-value problem (5) for small \( \eta \), which is known as the Blasius series, i.e.
\[
f(\eta) = \sum_{n=0}^{\infty} (-1)^n C_n [f^*(0)]^{n+1} (3n+2)! \eta^{3n+2}
\]
where the modern value of \( f^*(0) \) is 0.4695999 [14], and the coefficients \( C_n = 1 \); 1; 11; 375; 27579; 3817137; 856874115; 298013975; and so on. Weyl [2] has shown that these coefficients are given indeoinitely by
\[
C_n = \sum_{i=0}^{n+1} C_i C_{n+1-i} \left( \frac{3n-1}{3i} \right)
\]
However, Weyl [2] pointed out that the power series (6) has a finite radius of convergence, somewhere between 3.37 and 5.03. The estimate value of the radius of convergence of the power series (6) can be found by using the Domb-Sykes plot. The results correspond to the Blasius equation (5) are presented in Section V.

#### B. Falkner-Skan equation

The Falkner-Skan equation is obtained by transforming (3) using the variables in (4). The free stream velocity was first found as \( u_0(x) \propto x^m \) by Falkner and Skan in 1931. By selecting \( u_0(x) = x^n \) and \( g(x) = [2x^{1-m} / (m+1)]^{1/2} \) in (4) and substituting them into (3), we get
\[
f^{m} + ff^{n} + \beta(1 - f'^2) = 0
\]
which is known as the Falkner-Skan equation. The physical relevant solutions of (7) exist for values of \( \beta \) in the range of \(-0.19884 < \beta < 2 \) [5]. Separation point arises when \( \beta = -0.19884 \), which is first obtained by K. Stewart in 1931.

The perturbation method is suitable to solve the Falkner-Skan equation because it contains a parameter \( \beta \), which can be designated as the perturbation quantity. Aziz and Na [12] used this method to find the values of \( f^*(0; \beta) \). The function \( f(\eta) \) is first expanded as series in \( \beta \) such that
\[
f(\eta; \beta) = \sum_{n=0}^{\infty} \beta^n f_n(\eta)
\]
Then by substituting the series and its derivatives into (7), we will get the following resulting sequence.

\[
O(\beta^0): \quad f^{m}_n + f_{n-1} f^{n-1}_n = 0 \quad , \quad f_n(0) = f'_n(0) = 0 \, , \quad f_n(\infty) = 1
\]

\[
O(\beta^1): \quad f^{m}_n + f_{n-1} f^{n-1}_n + \frac{n}{2} f'_n f^{n-1}_n = \sum_{k=0}^{n-1} f_{n-k} f^{k} \quad , \quad f_n(0) = f'_n(0) = 0 \, , \quad f_n(\infty) = 0 \ \text{for} \ n \geq 1
\]

The zero-order problem (8a) corresponds to the Blasius problem. On the other hand, the problems (8b) of \( O(\beta^1) \) for \( n \geq 1 \) result the sequence of linear differential equations. Hence, they can be solved numerically. Then, each equation of \( O(\beta^1) \) will give the result of function \( f_n(\eta) \). Therefore the solution of (7) can be written as
\[
f(\eta; \beta) = f_0(\eta) + \beta f_1(\eta) + \beta^2 f_2(\eta) + \cdots
\]
or, in the form of velocity
\[
\frac{\partial \psi}{\partial y} = f_0(\eta) + \beta f_1(\eta) + \beta^2 f_2(\eta) + \cdots
\]
$u(x)/u(x) = f'(\eta, \beta) = f_0'(\eta) + \beta f_1'(\eta) + \beta^2 f_2'(\eta) + \ldots$ \hspace{1cm} (9)

The set of (8a) and (8b) is solved and the result of $f_n'(\eta)$ is presented in Section V.

IV. IMPROVEMENT OF SERIES

A. Domb-Sykes Plot

The radius of convergence $x_0$ of the power series

$$P(x) = \sum_n a_n x^n$$

may in principle be found by d’Alembert’s ratio test as

$$x_0 = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$ \hspace{1cm} (10)

Of course knowing only a finite number of the coefficients $a_n$, we cannot take the limit, but can only estimate it. Domb and Sykes [15] have introduced a method on estimating the value of radius of convergence graphically. Domb and Sykes have pointed out that it is more reliable to plot $a_n/a_{n-1}$ versus $1/n$, i.e. to bring $n \to \infty$ to the origin, rather than plotting $a_{n+1}/a_n$ versus $n$. The plot of $a_n/a_{n-1}$ versus $1/n$ is known as Domb-Sykes plot. From this plot, the radius of convergence $x_0$ can be estimated by linear extrapolation, which is the reciprocal value of the interception at $1/n = 0$. Therefore, (10) can also be written as

$$x_0 = \lim_{1/n \to 0} \left| \frac{a_n}{a_{n-1}} \right|$$ \hspace{1cm} (11)

The reason is that, for the simple singular function

$$P(x) = \text{constant} \times \begin{cases} (x_0 + x)^\alpha, & \alpha \not\in \mathbb{N} \\ (x_0 + x)^\alpha \log(x_0 + x), & \alpha \in \mathbb{N} \end{cases} \hspace{1cm} (12a)$$

$$P(x) = \begin{cases} (x_0 + x)^\alpha, & \alpha \not\in \mathbb{N} \\ (x_0 + x)^\alpha \log(x_0 + x), & \alpha \in \mathbb{N} \end{cases} \hspace{1cm} (12b)$$

the inverse ratio of coefficient is exactly linear in $1/n$, thus [13]

$$\frac{a_n}{a_{n-1}} = -\frac{1}{x_0} \left( 1 + \frac{\alpha}{n} \right)$$ \hspace{1cm} (13)

B. Shanks Transformation

Shanks [16] has proposed a family of nonlinear transformations to accelerate the convergence of slowly convergent and divergent series. Out of a number of transformations that are available, we select the simplest two designated by Shanks as $e_1$ and $e_n^m$. If three partial sums $S_{n-1}$, $S_n$ and $S_{n+1}$ of a series are known, then

$$e_1(S_n) = \frac{S_{n+1}S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n}$$

The second transformation $e_n^m$ is the $m$th iteration of $e_1$. For example, $e_1^2$ is obtained by treating the sequence $e_1(S_n)$ as partial sums, and so on [17]. This approach was inspired from Van Dyke [13] who used it to improve the series for ground-state energy of anharmonic oscillator.

V. NUMERICAL RESULTS & DISCUSSION

A. Radii of Convergence

To find the radius of convergence of the Blasius series (6), we may rearrange the series as

$$f(\eta) = \eta^2 \sum_{n=0}^{\infty} a_n \eta^{3n}$$ \hspace{1cm} (14)

where $a_n = (-1)^n C_n (f^*(0))^{n+1}/(3n+2)!$. Expression (14) is a formal power series of $\eta^2$, aside from a multiplicative term $\eta$. Hence, we can plot the values of $a_n/a_{n-1}$ versus $1/n$, where

$$\frac{a_n}{a_{n-1}} = -\frac{f^*(0)}{3n(3n+1)(3n+2)} C_n$$

The Domb-Sykes plot of the Blasius series (6) is shown in Figure 1, where the radius of convergence can be estimated as $1/0.01535 = 65.13385$. Therefore, we can say that the series (14) converges for $\eta$ such that $|\eta| < 65.13385$ or $|\eta| < 4.0235$, which is agreeable with the range given by Weyl [2].

![Figure 1. Domb-Sykes plot of the Blasius series (6)](image-url)
Since the radius of convergence of series (15) is estimated as 0.2, we notice that the series (15) is only valid for $|\beta| \leq 0.2$. However, in this case, $\beta$ is in the range of $-0.19884 < \beta < 2$. Therefore, Aziz and Na [12] have exploited the Shanks transformation to extend the range of validity of series (15). Using the same method, we will solve the velocity profiles of the Falkner-Skan equation (7) in the form of (9) for some values of $\beta$.

### B. Improvement of Series

The set of (8) has been solved using forth order Runge-Kutta method for the functions $f_\eta(\eta)$ and $f'_\eta(\eta)$. The expression $f'_\eta(\eta)$ is more significant than $f_\eta(\eta)$ because it corresponds to the velocity of the fluid as expressed in (9). The solution of $f'_n(\eta)$ is shown in Figure 3(a)-(e) for $n=0,1,2,3$ and 4.

From the five terms of $f'_\eta(\eta)$, the series of $f'(\eta;\beta)$ of (9) can be obtained. Table 1 demonstrates the finding of $f'(\eta;\beta)$ which can be written as

$$u/u_c(x) = f'(\eta;\beta) \approx e^2(S_2),$$

where $e^2(S_2) = \frac{e_1(S_2)e_1(S_1) - e_1(S_2)^2}{e_1(S_1) + e_1(S_2) - 2e_1(S_2)}$.

The last expression is the approximate solution of the Falkner-Skan equation (7). Even though only five terms of the perturbation series are considered, the result shows good agreement with the solution obtained by Hartree [5], as shown in Figure 4. This reflects the remarkable advantage in using the Shanks transformation in improving the perturbation series.
VI. CONCLUSION

The study on the Blasius and Falkner–Skan equations arise from the laminar boundary layer problems exhibiting similarity has been presented. In this paper, the solutions of the Blasius equation and Falkner-Skan equation is presented in form of formal power series and perturbation series respectively. The solution in terms of perturbation series is improved further using Shanks transformation. Domb-Sykes plot is used to examine the radii of convergence of the Blasius series and the perturbation series solution of the Falkner-Skan. The results which have been obtained show good agreements with previous results reviewed.

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Table I

<table>
<thead>
<tr>
<th>n</th>
<th>( S_n )</th>
<th>( \varepsilon_i )</th>
<th>( \varepsilon_i^2 )</th>
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<tr>
<td>0</td>
<td>( f'_0 )</td>
<td></td>
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</tr>
<tr>
<td>1</td>
<td>( f'_0 + \beta f''_0 )</td>
<td>( \varepsilon_i(S_i) = \frac{S_iS_0 - S_0^2}{S_0 + S_0 - 2S_0} )</td>
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</tr>
<tr>
<td>2</td>
<td>( f'_0 + \beta f''_0 + \beta^2 f'''_0 )</td>
<td>( \varepsilon_i(S_i) = \frac{S_iS_0 - S_0^2}{S_0 + S_0 - 2S_0} )</td>
<td>( \varepsilon_i(S_i)\varepsilon_i(S_i) - \varepsilon_i(S_0)^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( f'_0 + \beta f''_0 + \beta^2 f'''_0 + \beta^3 f^{iv}_0 )</td>
<td>( \varepsilon_i(S_i) = \frac{S_iS_0 - S_0^2}{S_0 + S_0 - 2S_0} )</td>
<td>( \varepsilon_i(S_i)\varepsilon_i(S_i) - \varepsilon_i(S_0)^2 ) + 2( \varepsilon_i(S_0) )</td>
</tr>
<tr>
<td>4</td>
<td>( f'_0 + \beta f''_0 + \beta^2 f'''_0 + \beta^3 f^{iv}_0 + \beta^4 f^{v}_0 )</td>
<td>( \varepsilon_i(S_i) = \frac{S_iS_0 - S_0^2}{S_0 + S_0 - 2S_0} )</td>
<td>( \varepsilon_i(S_i)\varepsilon_i(S_i) - \varepsilon_i(S_0)^2 ) + 2( \varepsilon_i(S_0) ) + ( \varepsilon_i(S_0) )</td>
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Figure 3(a)-(e) Solutions of \( f'_n(\eta) \) for \( n = 0,1,2,3 \) and 4 versus \( \eta \) of the resulting sequence (8).
REFERENCES


