The stochastic monetary model with delay

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Abstract: This paper addresses an autonomous stochastic system with delay which describes a monetary system involving interest rate, investment demand and price index. We analyze the deterministic model and the stochastic model with perturbation. For the stochastic system with delay we identify the differential equations for the mean values as well as for the mean square value. The last part of the paper includes numerical simulations and conclusions.

Key-Words: Monetary system, deterministic model with delay, stochastic delay system.

1 Introduction

Systems with delay are ubiquitous in nature and life. Uncertainty can be incorporated either as an expression of our lack of precise knowledge or as a true driving force. In the latter case it is useful to model, as the system by a stochastic or noise driven model, as in familiar in the finite dimensional case. In recent literature, it has been demonstrated that information delay makes dynamic economic models unstable. In situations where delays are important, models with stochastic perturbation are framed by stochastic differential delay equations. In this paper, we will investigate the effects of random perturbation for a delayed model, a monetary system composed of product, money, bound and labor force and analyze the steady state of model with stochastic perturbation. The similar analysis were made in [7], [8], [9].

The main purpose of this paper is to prove that different time lags can generate different stochastic dynamics. To this end we construct a stochastic delay monetary model and analyze the mean value in the steady state, the mean square and the dispersion of the system variables.

The rest of the paper develops as follows. In section 2, we describe a dynamic deterministic monetary system with delay and the model with stochastic perturbation. In section 3, we analyze the linearized of the deterministic model, establishing the existence conditions for the delay parameter’s values, for which the monetary system has a Hopf bifurcation. In section 4, we analyze the differential equations systems described by the variables’ mean and the mean square. In section 5, we present a few numerical simulations. Finally, concluding remarks will be given in section 6.

2 The mathematic deterministic and stochastic model with delay

The monetary system composed of product, money, bound and labor force is shaped by the differential equation system [1]:

\[
\begin{align*}
\dot{x}_1(t) &= -ax_1(t) + x_3(t) + x_1(t)x_2(t) \\
\dot{x}_2(t) &= 1 - bx_2(t) + x_1(t)^2 \\
\dot{x}_3(t) &= -x_1(t) - cx_3(t)
\end{align*}
\]

(1)

The variables \(x_1\), \(x_2\) and \(x_3\) denote the interest rate, the investment demand, and the price index, respectively. The positive parameters \(a\), \(b\) and \(c\) denote the saving amount, the per-investment cost, and the demand elasticity of commercials, respectively. The factors that generate the change of the interest rate \(x_1\) mainly come from two aspects: the price index and the savings amount. The changing rate of \(x_3\) is determined by the benefit rate of investment, the feedback of the investment...
demand, and the interest rate. The change of the price index \((x_1)\) is controlled by real interest rate and price index.

Chen [2] investigated the bifurcation behavior occurring in the system (1). In [5] it is demonstrated that the system (1) is not smoothly equivalent to the generalized Lorentz canonical form.

If we take into account that the interest rate, the investment demand and the price index in the moment \(t - \tau\), \(\tau > 0\) is \(x_1(t - \tau), x_2(t - \tau), x_3(t - \tau)\), the monetary system with delay is:

\[
x_1(t) = -ax_1(t) - a_1 x_1(t - \tau) + x_2(t) + x_1(t)x_2(t)
\]

\[
x_2(t) = 1 - bx_2(t) - b_1 x_2(t - \tau) - x_1(t)^2
\]

\[
x_3(t) = -x_1(t) - c x_3(t) - c_1 x_3(t - \tau)
\]

with the initial condition:

\[
x_1(\theta) = \phi_1(\theta), x_2(\theta) = \phi_2(\theta), x_3(\theta) = \phi_3(\theta), \theta \in [-\tau, 0].
\]

The model (2) represents a system of differential equations with delay.

**Proposition 1:**

The equilibrium of the system (2) is \((x_{10}, x_{20}, x_{30})\) where:

\[
x_{10} = 0, x_{20} = \frac{1}{b + b_1}, x_{30} = 0
\]

and if \(c + c_1 - b - b_1 > (a + a_1)(b + b_1)(c + c_1)\), then \((x_{11}, x_{21}, x_{31}), (x_{12}, x_{22}, x_{32})\) with:

\[
x_{11} = -c_1 x_{31}, x_{21} = \frac{1}{b + b_1}(c + c_1)x_{31}^2,
\]

\[
x_{31} = \frac{1}{c + c_1} \sqrt{c + c_1 - b - b_1 -(a + a_1)(b + b_1)(c + c_1)}
\]

\[
x_{12} = -c_1 x_{32}, x_{22} = \frac{1}{b + b_1}(c + c_1)x_{32}^2,
\]

\[
x_{32} = -\frac{1}{c + c_1} \sqrt{c + c_1 - b - b_1 -(a + a_1)(b + b_1)(c + c_1)}
\]

Let the probability space \((\Omega, F, P)\) be given, and \(w(t) \in \mathbb{R}\) be a scalar Wiener process defined on \(\Omega\) having independent stationary Gaussian increments with \(w(0) = 0, E(w(t) - w(s)) = 0\) and \(E(w(t) - w(s)) = \min(t, s)\). The symbol \(E\) denotes the mathematical expectation [3]. The sample trajectories of \(w(t)\) are continuous, nowhere differentiable and have infinite variation on any finite time interval.

What we are interested to know is the effect of the noise perturbation on the equilibrium \((x_{10}, x_{20}, x_{30})\). The stochastic perturbation of (1) given by the form of a stochastic differential equation with delay is given by:

\[
dx_1(t) = (-ax_1(t) - a_1 x_1(t - \tau) + x_2(t) + x_1(t)x_2(t))dt + \sigma_1(x_1(t) - x_{10})dw(t)
\]

\[
dx_2(t) = (1 - bx_2(t) - b_1 x_2(t - \tau) - x_1(t)^2)dt + \sigma_2(x_2(t) - x_{20})dw(t)
\]

\[
dx_3(t) = (-x_1(t) - c x_3(t) - c_1 x_3(t - \tau))dt + \sigma_3(x_3(t) - x_{30})dw(t)
\]

Linearizing (5) around the equilibrium yields the linear stochastic differential delay equation:

\[
dy(t) = (Ay(t) + By(t - \tau))dt + Cy(t)dw(t)
\]

where \(y(t) = (y_1(t) + y_2(t)dt + y_3(t))^T\) and

\[
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}
\]

where,

\[
a_{11} = -a, \quad a_{12} = 0, \quad a_{13} = 1, \quad b_{11} = -a_1,
\]

\[
a_{21} = 0, \quad a_{22} = -b, \quad a_{23} = 0, \quad b_{22} = -b_1,
\]

\[
a_{31} = -1, \quad a_{32} = 0, \quad a_{33} = -c, \quad b_{33} = -c_1.
\]

### 3 The analysis of a linear differential delay equation

The linear differential equation with delay is given by:

\[
y(t) = Ay(t) + By(t - \tau)
\]

From (9) and (7) we obtain:

**Proposition 2:**

1. The characteristic function of the system (9) is:

\[
h(\lambda, \tau) = (\lambda^2 + \lambda(a + c) + ac + 1 + (\lambda a_1 + c_1) + a_1 q_2 c e^{-\lambda \tau} + a q_1 c e^{-\lambda \tau} + b q_1 c e^{-\lambda \tau})
\]

2. If \(a_1 = b_1 = c_1 = 0\) and \(\tau = 0\), the characteristic equation \(h(\lambda, \tau) = 0\) has roots with negative real parts. The solutions of the system (1) are asymptotically stable.

3. If \(a_1 = c_1 = 0\), \(b_1 > b\) and \(\tau \in (0, \tau_0)\) the characteristic equation \(h(\lambda, \tau) = 0\) has roots...
have negative real parts, where \( \tau_0 \) is given by:
\[
\tau_0 = \frac{1}{\sqrt{b_1^2 - b^2}} \arccos \left( -\frac{b}{b_1} \right) \tag{11}
\]
If \( \tau = \tau_0 \) system (1) has a limit cycle with a period of oscillation given by:
\[
T_0 = \frac{2\pi}{\sqrt{b_1^2 - b^2}} \tag{12}
\]

4. If \( c_1 = 0, b_1 = 0, a^2 + c^2 > 2 \), \( a_1 \in \left( \frac{1}{a} \sqrt{a^2 + c^2 - 2} \right) \) and \( \tau \in (0, \tau_1) \) the characteristic equation \( h(\lambda, \tau) = 0 \) has roots with negative real parts, where \( \tau_1 \) is given by:
\[
\tau_1 = \frac{1}{a_0} \arctg \left( \frac{a(1 - \omega^2 - c^2)}{\omega(\omega^2 + c)} + c \right) \tag{13}
\]
and \( a_0 \) is the positive root of the equation:
\[
\omega^4 + (a^2 + c^2 - a_1^2 - 2)\omega^2 + (1 + ac)^2 - a_0^2c^2 = 0. \tag{14}
\]

5. If \( a_1 = 0, b_1 = 0, a^2 + c^2 > 2 \), \( c_1 \in \left( \frac{1}{a} \sqrt{a^2 + c^2 - 2} \right) \) and \( \tau \in (0, \tau_2) \) the characteristic equation \( h(\lambda, \tau) = 0 \) has roots with negative real parts, where \( \tau_2 \) is given by:
\[
\tau_2 = \frac{1}{a_0} \arctg \left( \frac{a(1 - \omega^2 + a^2)}{c(\omega^2 + a^2) + a} \right) \tag{15}
\]
and \( a_0 \) is the positive root of the equation:
\[
\omega^4 + (a^2 + c^2 - c_1^2 - 2)\omega^2 + (1 + ac)^2 - a_0^2c^2 = 0. \tag{16}
\]

**Proof:**

1. The characteristic function of (9) is:
\[
h(\lambda, \tau) = \text{det}(\lambda I - A - B e^{-\lambda \tau}) \tag{17}
\]
From (16) with (17) and (18), we get formula (10).

2. Since \( a_1 = c_1 = 0 \) and \( \tau = 0 \), the characteristic equation is:
\[
(\lambda^2 + \lambda(a + c) + ac + 1)(\lambda + b) = 0 \tag{18}
\]
Since \( a, b, c \) are positive numbers, it ensues that the equation (17) has roots with negative real parts.

3. Since \( a_1 = c_1 = 0 \) and \( b_1 > 0 \), the characteristic equation is:
\[
(\lambda^2 + \lambda(a + c) + ac + 1)(\lambda + b + b_1 e^{-\lambda \tau}) = 0
\]
It is well known [4], [6] that the necessary and sufficient condition for \( \text{Re}(\lambda) < 0 \) for the equation:
\[
\lambda + b + b_1 e^{-\lambda \tau} = 0
\]
is given by \( 0 < \tau < \tau_0 \), where \( \tau_0 \) is given by (11).

The period \( T_0 \) of oscillations is given by (12). For \( \tau = \tau_0 \) the normal form for the system (2) is obtained using the method from [3], [4].

### 3 The analysis of the linear differential stochastic equation with delay

Let \( Y(t) \) be the fundamental solution of the system (9). The solution of (5) is a stochastic process given by:
\[
y(t, \phi) = y(\phi(t)) + \int_0^t Y(t-s) C y(t-\tau, \phi) dw(s) \tag{19}
\]
where \( y(\phi(t)) \) is the solution given by:
\[
y(\phi(t)) = Y(t) \phi(0) + \int_0^t Y(t - s) \phi(s) ds \tag{20}
\]
and \( \phi: [-\tau, 0] \rightarrow R^3 \) is a family of continuous functions.

The existence and uniqueness theorem for the stochastic differential delay equations has been established in [3].

The solution \( y(t, \phi) \) is a stochastic process with distribution at any time \( t \) determined by the initial function \( \phi(\theta) \). From the Chebyshev inequality, the possible range of \( y \), at time \( t \) is “almost” determined by its mean and variance at time \( t \). Thus, the first and second moments of the solutions are important for investigation the solution behavior.

We have used \( E \) to denote the mathematical expectation and we denote \( y(t, \phi) \) by \( y(t) \).

**Proposition 3:**

The moments of the solution of (6) is given by:
\[
\frac{dE(y(t))}{dt} = AE(y(t)) + BE(y(t - \tau)) \tag{21}
\]

The mean of the solution for (6) behaves precisely like the solution of the unperturbed deterministic equation (9).

The proof results from taking into account the mathematical expectation of both sides of (6) as well as the properties of the Ito calculus.

To examine the stability of the second moment of \( y(t) \) for linear stochastic differential delay equation (6) we use Ito’s rule to give the stochastic differential of \( y(t) y(t)^T \) where \( y(t) = (y_1(t), y_2(t), y_3(t))^T \):
\[ \frac{d}{dt} E(y(t)^T(t)) = E(dy(t)^T(t)) + y(t)dy(t) + C_3(y(t)^T(t)C) \]
\[ = E(Ay(t)^T(t)) + y(t)dt + B(t) - \tau y(t)^T(t) + (22) \]
\[ + y(t)dt - \tau B(t) + C(y(t)^T(t)) \]

Let \( R(t,s) = E \{ y(t)^T(s) \} \) be the covariance matrix of the process \( y(t) \) so that \( R(t,t) \) satisfies:
\[ \dot{R}(t,t) = AR(t,t) + R(t,t)A^T + BR(t-t) + CR(t,t)C \]

**Proposition 4:**
1. The differential system (22) is given by:
\[ \dot{R}_{11}(t,t) = (2a_{11} + \sigma_1^2)R_{11}(t,t) + 2b_1 R_{11}(t,t-\tau) + \]
\[ + 2a_{12} R_{12}(t,t) + 2a_{13} R_{13}(t,t) \]
\[ \dot{R}_{22}(t,t) = (2a_{22} + \sigma_2^2)R_{22}(t,t) + 2b_2 R_{22}(t,t-\tau) + \]
\[ + 2a_{21} R_{11}(t,t) + 2a_{23} R_{23}(t,t) \]
\[ \dot{R}_{33}(t,t) = (2a_{33} + \sigma_3^2)R_{33}(t,t) + 2b_3 R_{33}(t,t-\tau) + \]
\[ + 2a_{31} R_{13}(t,t) + 2a_{32} R_{23}(t,t) \]
\[ \dot{R}_{12}(t,t) = a_{12}(t,t) + a_{12} R_{12}(t,t) + \]
\[ + (a_{11} + a_{22} + \sigma_2^2)R_{12}(t,t) + \]
\[ + b_1 R_{12}(t,t-\tau) + b_2 R_{22}(t,t-\tau) + \]
\[ + a_{23} R_{23}(t,t) + a_{32} R_{32}(t,t) \]
\[ \dot{R}_{13}(t,t) = a_{13}(t,t) + a_{13} R_{13}(t,t) + \]
\[ + (a_{11} + a_{33} + \sigma_3^2)R_{13}(t,t) + \]
\[ + b_1 R_{13}(t,t-\tau) + b_2 R_{23}(t,t-\tau) + \]
\[ + a_{32} R_{32}(t,t) + a_{32} R_{23}(t,t) \]
\[ \dot{R}_{23}(t,t) = a_{23}(t,t) + a_{23} R_{23}(t,t) + \]
\[ + (a_{21} + a_{32} + \sigma_3^2)R_{23}(t,t) + \]
\[ + b_2 R_{23}(t,t-\tau) + b_3 R_{33}(t,t-\tau) \]

2. The characteristic function of (23) is given by:
\[ h(\lambda, \tau) = \det \begin{pmatrix} 2I - A_1 - B_1 e^{-\lambda \tau} & A_2 \\ A_1 & 2I - A_2 - B_2 e^{-\lambda \tau} \end{pmatrix} \]

\[ \text{where,} \]
\[ A_1 = \begin{pmatrix} a_{11} + \sigma_1^2 & 0 & 0 \\ 0 & 2a_{22} + \sigma_2^2 & 0 \\ 0 & 0 & 2a_{33} + \sigma_3^2 \end{pmatrix} \]
\[ A_2 = \begin{pmatrix} 2a_{12} & 2a_{13} & 0 \\ 0 & 2a_{21} & 2a_{23} \\ 0 & 2a_{31} & 2a_{32} \end{pmatrix} \]
\[ B_1 = \begin{pmatrix} 2b_1 & 0 & 0 \\ 0 & 2b_1 & 0 \\ 0 & 0 & 2b_3 \end{pmatrix} \]
\[ B_2 = \begin{pmatrix} b_{11} + b_{22} & 0 & 0 \\ 0 & b_{11} + b_{33} & 0 \\ 0 & 0 & b_{22} + b_{33} \end{pmatrix} \]

**Proof:**
1. The system (23) result from (22) by taking into account that \( R_{ij}(t,t) = R_{ij}(t,t) , \ i, j = 1,2. \)
2. Let \( R_{ij}(t,s) = e^{\lambda(t+s)}K_{ij} , \ i, j = 1,2. \) where \( K_{ij} \) is a constant. Replacing \( R_{ij}(t,s) \) in (23) and setting the condition that the system we obtain should accept results different than 0, we get \( h_s(\lambda, \tau) = 0. \)

The characteristic function (21) with the coefficients given by (7) is:
\[ h_s(\lambda, \rho) = h_s(\lambda, \tau)h_s(\lambda) \]

where
\[ h_s(\lambda, \rho) = (2\lambda - \sigma_1^2 - 2\sigma_2^2)(2\lambda - \sigma_2^2 - 2\sigma_3^2) \]
\[ (2\lambda - \sigma_3^2 - 2\sigma_1^2)(2\lambda - \sigma_2^2 - 2\sigma_3^2) \]
\[ (2\lambda - \sigma_3^2 - 2\sigma_1^2)(2\lambda - \sigma_3^2 - 2\sigma_1^2) \]

Calculating the stability of the second moment is done by analyzing the roots of the characteristic equation \( h_s(\lambda, \tau) = 0. \)

Because the equation \( h_s(\lambda, \tau) = 0 \) encompasses the functions \( e^{-\lambda \tau} \) and \( e^{-2\lambda \tau} \), the study of the roots is done for cases in which \( a_1 = b_1 = 0 \) and \( b_1 = c_1 = 0. \)

### 5 Numerical simulations

For the numerical simulation, the following values were taken into consideration: \( a_2 = 2, b = 0.15, c = 1.5, a_1 = 1, b_1 = 2, c_1 = 1, \sigma_1 = 2.5, \sigma_2 = 0.65, \sigma_3 = 2. \) The equilibrium point is \( x_{10} = 0, x_{20} = 0.465, x_{30} = 0. \) For \( \tau = 0 \) it ensues that \( h(\lambda, 0) = 0, \) where \( h(\lambda, 0) \) is given by (9), and has roots with negative real parts. Thus, the equilibrium is asymptotically stable. The mean values of the interest rate, the investment demand and the price index are asymptotically stable.

The roots of the equation \( h(\lambda, 0) = 0, \) where \( h(\lambda, 0) \) is given by (27) have negative real parts.
Thus the mean square values, the interest rate variance, the investment demand and the price index are asymptotically stable.

For $\tau = 3$, fig.1, fig.2, fig.3 present the interest rate variations $(n, x_1(n, \omega))$, the investment demand $(n, x_2(n, \omega))$, and the price index $(n, x_3(n, \omega))$. The figures Fig.4, Fig.5, Fig.6, present $(n, E(x_1(n, \omega)))$, $(n, E(x_2(n, \omega)))$, $(n, E(x_3(n, \omega)))$. The figures Fig. 7, fig.8, fig.9 show the variances of the variable $y_1(t)$, $y_2(t)$, $y_3(t)$. 

Fig.1 The orbit $(n, x_1(n, \omega))$

of the interest rate

Fig.2 The orbit $(n, x_2(n, \omega))$

of the investment demand

Fig.3 The orbit $(n, x_3(n, \omega))$

of the price index

Fig.4 The orbit $(n, E(x_1(n, \omega)))$

Fig.5 The orbit $(n, E(x_2(n, \omega)))$

Fig.6 The orbit $(n, E(x_3(n, \omega)))$

Fig.7 The plot $V(y_1(t, \omega))$
References:

6 Conclusion
In this paper, we have examined a nonlinear stochastic monetary system with delay. We have analyzed the deterministic model and the stochastic system with perturbations. For the numerical values of the saving amount, investment cost and demand elasticity of commercials, respectively, we have established with the help the numerical simulations that the system’s solution is asymptotically stable. At the same time, we have established the mean and mean square values. The presented analysis can be employed for other economic models.

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