On the Design of 2-D Inverse and 2-D Wiener Filters

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Abstract **— In this paper, the problem of 2-D (two-dimensional) inverse and 2-D Wiener filtering is studied. 2-D Inverse and Wiener filters are designed. For the 2-D Wiener Filter design appropriate 2-D IIR Notch filters must be placed in cascade before the Wiener Filter itself.**

Keywords **— 2-D Filters, 2-D Inverse Filters, 2-D Wiener Filters, Multidimensional Systems, Multidimensional Filters, Stochastic Signals, Notch Filters, Power Spectral Density.**

I. Introduction

Image filtering and image restoration occurs in almost all the image processing applications since any image acquired by optical or digital means is likely to be degraded by the sensing environment. This degradation is due to blur, distortion, vignetting (light falloff), lateral chromatic aberration, lens flare, veiling glare and other forms of noise. From a signal processing standpoint, blurring due to linear motion in a photograph is the result of poor sampling. All this forms of noise can be considered as additive noise degrading the 2-D signal (image). On the other hand, Wiener filtering is a standard technique of signal processing that has been applied also to 2-D signals with success. The solution of the Wiener filtering is known [1], however the practical design of the Wiener filter appears to have some problems. On the other hand, in the theoretical case where no noise exists, our Wiener Filter yields an Inverse filter. In general the problem of 2-D filter design has received considerable attention during the recent years, due to its numerous applications in Biomedical Signal Processing, Digital Imaging, Radar Image Analysis, Seismic Data Processing, Geophysical Signal, Remote Sensing, Petroleum Research, Computer Vision, Sonar, X-Rays enhancement etc…[2]÷[19]. The concept of 2-D Stability has also received considerable attention [20]÷[26].

Suppose that our 2-D stochastic signal $x(n_1,n_2)$ has (PSD) Power Spectral Density S_{xx}, where $S_{xx}=\int_{\tau=-\infty}^{\tau=\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$ i.e. the Fourier Transform of the autocorrelation of the input Signal. Then if our 2-D linear, shift invariant system has transfer function $H\left(z_1^{-1},z_2^{-1}\right)$, the PSD of the output $y(n_1,n_2)$ defined as $S_{yy}=\int_{\tau=-\infty}^{\tau=\infty} R_{yy}(\tau)e^{-j\omega\tau}d\tau$ fulfils the relation $S_{yy}=\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}}\cdot S_{xx}$ The last equation is correct provided that no noise has been added in the output of our system. In this case, one can reconstruct the input signal $x(n_1,n_2)$ using the output signal by using a so called inverse filter with transfer function

$$G\left(z_1^{-1},z_2^{-1}\right)=\begin{cases}H^{-1}\left(z_1^{-1},z_2^{-1}\right), H^{-1}\left(z_1^{-1},z_2^{-1}\right)\neq 0\\ 0, otherwise\end{cases} \quad (1)$$

In practice, we use $G\left(z_1^{-1},z_2^{-1}\right)=H^{-1}\left(z_1^{-1},z_2^{-1}\right)$ if $\left|H\left(e^{j\omega_1},e^{j\omega_2}\right)\right|>\varepsilon$ and $G\left(z_1^{-1},z_2^{-1}\right)=0$ if $\left|H\left(e^{j\omega_1},e^{j\omega_2}\right)\right|\leq\varepsilon$ where ε is a small positive number.

In the case of additive noise in our system, we can reconstruct optimally the original signal by placing a filter in the output $y(n_1,n_2)$ of our system (Fig.1) with transfer function

$$G\left(z_1^{-1},z_2^{-1}\right)=\frac{H(z_1^{-1},z_2^{-1})^*_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}}\cdot S_{xx}}{\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}}\cdot S_{xx}+S_{NN}} \quad (2)$$

$H(z_1^{-1},z_2^{-1})^*_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}}$ is the complex conjugate of $H(z_1^{-1},z_2^{-1})_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}}$ while S_{NN} is the PDF of the additive Noise $N(n_1,n_2)$. Equation (2) gives the so-called 2-D Wiener Filter, a special case of which is the inverse filter of (1) if $S_{NN}=0$. Although the inverse and the Wiener filter are well known their implementation seems to present some difficulties. Our methodology will be described in Section II, first for Inverse filters and second for Wiener Filters. Finally, there is a conclusion.

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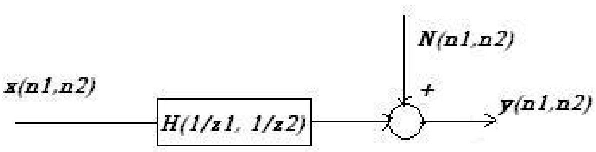

Fig.1. Additive noise

II. DESIGN OF 2-D INVERSE AND 2-D WIENER FILTERS

II.a 2-D Inverse Filter

For the 2-D Inverse filter, we assume that $H\left(e^{j\omega_1}, e^{j\omega_1}\right) \neq 0$, except some possible isolated points $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...(\alpha_m,\beta_m)$ of the 2-D complex plane for which $H\left(e^{j\omega_1}, e^{j\omega_1}\right) = 0$. This is reasonable, otherwise it would be impossible to reconstruct the original 2-D signal, if we had a subset S of $\omega_1,\omega_2 \subset (-\pi,\pi)\times(-\pi,\pi)$ with $H\left(e^{j\omega_1}, e^{j\omega_1}\right) = 0$ and $\mu(S) \neq 0$. The measure $\mu(S)$ here is the **area** of the subset S in the plane of ω_1,ω_2.

Suppose that $H\left(e^{j\omega_1}, e^{j\omega_1}\right) = \dfrac{A\left(e^{j\omega_1}, e^{j\omega_1}\right)}{B\left(e^{j\omega_1}, e^{j\omega_1}\right)}$ and assume $B\left(e^{j\omega_1}, e^{j\omega_1}\right) \neq 0$, except the points of $(\omega_1,\omega_2) = (\alpha_1,\beta_1),(\alpha_2,\beta_2),...(\alpha_m,\beta_m)$

Now the inverse filter is given as follows:

$$G\left(e^{j\omega_1}, e^{j\omega_1}\right) = \frac{B\left(e^{j\omega_1}, e^{j\omega_1}\right)}{A\left(e^{j\omega_1}, e^{j\omega_1}\right)} \qquad (3)$$

II.b 2-D Wiener Filter

For the 2-D Wiener Filter we have

$$G\left(z_1^{-1}, z_2^{-1}\right) = \frac{H(z_1^{-1},z_2^{-1})^{*}_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx}}{\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN}}$$

We demand now $\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN} \neq 0$ except some possible isolated points $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...(\alpha_m,\beta_m)$

So, if we have frequencies $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...(\alpha_m,\beta_m)$ such that $\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN} = 0$, we must place the appropriate 2-D (IIR) notch filters, in cascade, before our Wiener Filter, in order to reject these frequencies (avoiding possible problem for the denominator of (2)).

One can see that if (α_1,β_1) is a solution of $\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN} = 0$ then the points $(-\alpha_1,\beta_1),(\alpha_1,-\beta_1),(-\alpha_1,-\beta_1)$ will also be solutions due to the symmetry of S_{xx},S_{NN}. Actually, because we assume stationary stochastic signals and stationary noise, we have:

$$S_{xx}(\omega_1,\omega_2) = S_{xx}(\omega_1,-\omega_2) = S_{xx}(-\omega_1,\omega_2) = S_{xx}(-\omega_1,-\omega_2)$$

and

$$S_{NN}(\omega_1,\omega_2) = S_{NN}(\omega_1,-\omega_2) = S_{NN}(-\omega_1,\omega_2) = S_{NN}(-\omega_1,-\omega_2)$$

The author has proposed recently a methodology for the first and second order 2-D Notch Filters, [27], which can be used here.

This kind of 2-D Notch filter can be designed, [27], as follows for the frequencies $(\pm\alpha_1,\pm\beta_1)$ that vanish the $\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN}$

$$H\left(z_1^{-1},z_2^{-1}\right) = K\frac{1^2+c^2+2c\cos(\omega_{10}T_1-\omega_{20}T_2)-2(z_1^{-1}+cz_2^{-1})(\cos(\omega_{10}T_1)+c\cos(\omega_{20}T_2))+(z_1^{-1}+cz_2^{-1})^2}{1^2+c^2+2c\cos(\omega_{10}T_1-\omega_{20}T_2)-2r(z_1^{-1}+cz_2^{-1})(\cos(\omega_{10}T_1)+c\cos(\omega_{20}T_2))+r^2(z_1^{-1}+cz_2^{-1})^2}$$

(4)

where $(\omega_{10},\omega_{20}) = (\alpha_1,\beta_1)$

$c = \dfrac{1-\lambda}{\lambda}$, $0<\lambda<1$, but with $\lambda \neq \dfrac{1}{2}$, T_1,T_2 sampling periods and $0 << r < 1$. K is a scaling factor such that the maximum gain of the filter to be equal to 1.

This 2-D Notch Filter can reject the frequencies $(\omega_{10},\omega_{20}) = (\alpha_1,\beta_1)$ and $(\omega_{10},\omega_{20}) = (-\alpha_1,-\beta_1)$, but not $(\omega_{10},\omega_{20}) = (\alpha_1,-\beta_1)$ and $(\omega_{10},\omega_{20}) = (\alpha_1,-\beta_1)$. So, for the pair $(\alpha_1,-\beta_1),(\alpha_1,-\beta_1)$ another 2-D Notch filter must be used.

Therefore it is necessary to put before the 2-D Wiener Filter, $2m_1$ appropriate notch filters where $m_1 = \dfrac{m}{4}$

Remark: For the inverse filters (case without noise), Notch filters do not need.

<u>Example of a 2-D Wiener Filter Design</u>

Suppose that $\left|H(z_1^{-1},z_2^{-1})\right|^2_{z_1^{-1}=e^{j\omega_1},z_2^{-1}=e^{j\omega_2}} \cdot S_{xx} + S_{NN} \neq 0$ except the points $(\frac{\pi}{2},\frac{\pi}{4})$, $(\frac{\pi}{2},-\frac{\pi}{4})$, $(-\frac{\pi}{2},\frac{\pi}{4})$, $(-\frac{\pi}{2},-\frac{\pi}{4})$

Then we will need only two 2-D Notch Filter, the first of them that can be designed by replacing in (4), $(\omega_{10},\omega_{20}) = (\frac{\pi}{2},\frac{\pi}{4})$ and the second one by replacing in (4), $(\omega_{10},\omega_{20}) = (-\frac{\pi}{2},\frac{\pi}{4})$

Select c = 2, $r = 0.9$. Consider also without loss of generality T_1, T_2 = 1. Then for the first 2-D Notch filter, one has:

$$H\left(z_1^{-1},z_2^{-1}\right) = K\frac{(z_1^{-1}+2z_2^{-1}+3j)(z_1^{-1}+2z_2^{-1}-3j)}{(rz_1^{-1}+r2z_2^{-1}+3j)(rz_1^{-1}+r2z_2^{-1}-3j)}$$

with $z_1^{-1} = e^{j\omega_1}$ $-\pi \le \omega_1 \le \pi$, $z_2^{-1} = e^{j\omega_2}$, $-\pi \le \omega_2 \le \pi$

Then magnitude response of the first filter is depicted in Fig.2, while the Group Delays of the first filter

$$\tau_1 = -\frac{\partial ArgH(j\omega_1, j\omega_2)}{\partial\omega_1},\ \tau_1 = -\frac{\partial ArgH(j\omega_1, j\omega_2)}{\partial\omega_2}$$

are depicted in Fig.3 and Fig.4

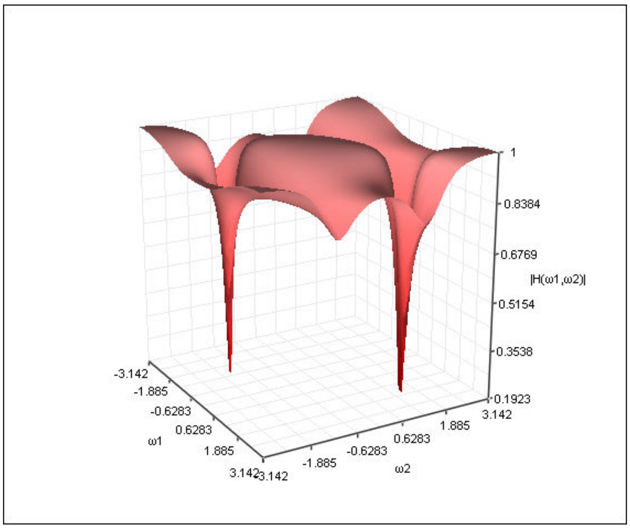

Fig.2

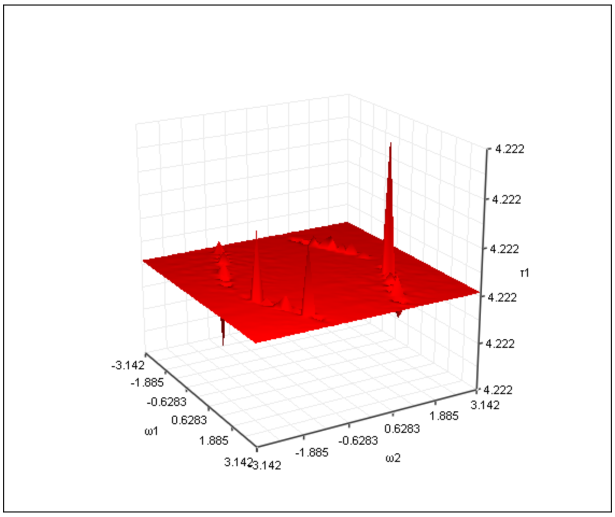

Fig.3

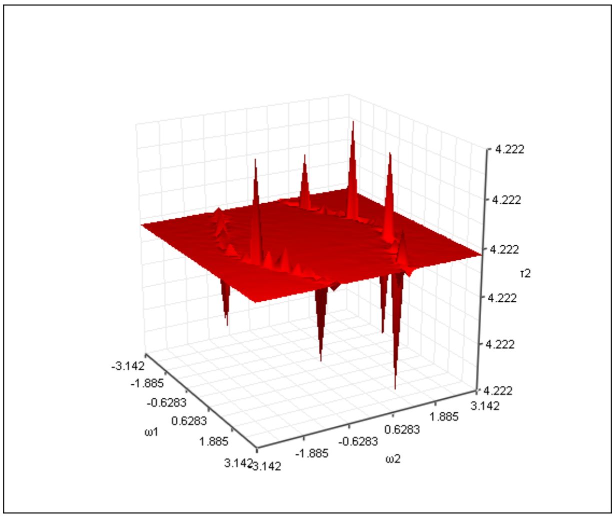

Fig.4

The design of the second 2-D Notch filter is quite similar and is omitted for the sake of brevity.

III. Conclusion

In this paper, we have examined the problem of the design of 2-D (Two-Dimensional) IIR Inverse and Wiener filters. For the 2-D Wiener Filter design appropriate 2-D IIR Notch filters must be placed in cascade before the Wiener Filter.

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