Existence of Shocklets in Supersonic Boundary Layer and Its Relation with 3-D Disturbances

Yihong Fang, Yifang Liu, and Xingfu Ye

Abstract—We present results from direct numerical simulation (DNS) of a supersonic boundary layer with Mach number 4.5. The main new phenomenon is the appearance of shocklets which form during the evolutionary process of three-dimensional (3-D) disturbances. A detection algorithm is developed to extract and locate the shocklet from the DNS fields. The relationship between the wave numbers and the emergence of shocklets were adequately investigated.

Keywords—shocklet; supersonic boundary layer; disturbance; second mode; first mode

I. INTRODUCTION

The boundary layer transition in supersonic and hypersonic flows is a crucial and perplexing problem in the design of aerodynamic vehicles. For instance, a good estimate of skin friction drag and wall heat transfer rate relies on an accurate prediction of the transition location.

For supersonic flat-plate boundary layer, the linear stability theory has long been adopted to study disturbance evolution [1,2]. Compared to incompressible boundary layers, the nonlinear stages of transition in a supersonic boundary layer remains poorly understand despite many theoretical study and numerical simulations during last two decades. It is clear that compressible flow admits a phenomena not observed in incompressible flow, namely the production of shocklets. The appearance of shocklets would make the flow field discontinuous, therefore the nonlinear theory which is good for incompressible flow, may not work for supersonic flow[3,4].

The fact that shocklets would be induced by disturbances in a two-dimensional (2-D) compressible mixing layers has been confirmed[5]. The existence of shocklets due to two-dimensional disturbances of supersonic boundary layer was also demonstrated [6,7]. A preliminary study of shocklets induced by three-dimensional (3-D) disturbances in a supersonic boundary layer has been carried out[8], but no conclusive result has been obtained. If shocklets would be induced, what would be the wave number and amplitude of disturbances when shocklets started to appear are problems need to be addressed.

In this paper, the nonlinear evolution of 3-D disturbance waves are investigated by direct numerical simulation in a supersonic boundary layer with Mach number 4.5. At a chosen location, disturbance waves were fed into the flow field. Some nonlinear characteristics of its evolution have been found. A well-proved algorithm of detecting shocklets from the DNS fields was extended in current research. The existence of shocklets has been proof-tested. Moreover, the effect of an important parameter, the spanwise wave number, which is an issue that has not been elaborated sufficiently in previous study is emphasized.

II. EQUATIONS AND NUMERICAL METHOD

The governing equations were Navier-Stokes equations in strong conservation form and an equation of state for perfect gas. All the quantities were made non-dimensional by referring to the displacement thickness at the entrance $\delta$, $U_e$, the velocity at infinity, $\rho_\infty$, $T_e$ and $\mu_\infty$, the density, temperature and viscosity respectively at the infinity. The Prandtl number $Pr=\mu c_p/\lambda$ was taken to be 0.72 in our computation, where $c_p$ is the specific heat of gas at constant pressure, $\gamma=P_\gamma-1$, $\gamma$ is the adiabatic exponent of the gas, $\gamma=1.4$.

The non-dimensional compressible Navier-Stokes equation in conservative form was

$$\frac{\partial UE}{\partial t} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = E - \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$$

where $U$ is the flux, $\partial E/\partial x$, $\partial F/\partial y$ and $\partial G/\partial z$ the convective terms, including pressure gradient; $\partial E_x/\partial x$, $\partial F_y/\partial y$ and $\partial G_z/\partial z$ the viscous terms, including heat conduction.

The above equations were solved numerically with compact finite difference scheme[9]. The convective terms were solved with a fifth order weakly upwind compact scheme proposed in equations (2).

$$2 \frac{\partial E_{n+1}^x}{\partial x} + 3 \frac{\partial E_{n}^x}{\partial x} = \frac{1}{12\Delta x} \left( -E_{n+2}^x + 12E_{n+1}^x + 36E_{n}^x - 44E_{n-1}^x - 3E_{n-2}^x \right)$$

$$3 \frac{\partial E_{n+1}^x}{\partial x} + 2 \frac{\partial E_{n}^x}{\partial x} = \frac{1}{12\Delta x} \left( -E_{n+2}^x - 12E_{n+1}^x - 36E_{n}^x + 44E_{n-1}^x + 3E_{n-2}^x \right)$$
The scheme for the viscous terms was a sixth order compact central difference scheme,
\[ 12 f_{i+1}^{n+3} + 36 f_{i+1}^{n+2} + 12 f_{i+1}^{n+1} = \frac{1}{\Delta x} \left(28(f_{i+1}^{n+3} - f_{i+1}^{n+1}) + (f_{i+1}^{n+2} - f_{i+1}^{n+2})\right) \tag{3} \]

The time marching is an explicit low-storage second-order Runge-Kutta technique. The Navier-Stokes characteristic boundary condition (NSCBC) proposed by Poinsot & Lele[10], was used at the exit and upper boundary, non-slip and adiabatic conditions were used for the wall. A period condition was employed in spanwise.

III. BASIC FLOW AND DISTURBANCES

A. Computational Domain

The numerical simulation for the evolution of 3-D disturbance was done for a boundary layer of a flat plate with Mach number 4.5. The parameters of the air corresponded to the condition at a height of 5000m. We took \( Re_x = 90000 \), the corresponding \( \delta = 1.38 \text{mm} \), where \( \delta \) was the displacement thickness at \( x^* = 1.7197 \text{m} \) from the leading edge according to the similarity solution. As \( \delta \) was taken as the reference length scale, then the computational domain started at \( x^* = 1246 \). The non-dimensional length of computational domain in \( y \) direction was 6, but non-uniform meshes were used. The coordinates of grid points were determined by \( y = 6(e^{\eta - 1})/(e^2 - 1) \), which transformed the interval \([0,1]\) of \( \eta \) into an interval \([0,6]\) of \( y \), and then \( \eta \) was evenly distributed in \([0,1]\), having totally 200 meshes. In \( z \) direction, the domain length was 6, but non-uniform meshes were used.

The isosonic contours of the final steady flow as shown in Fig.1(a) there was a narrow zone where the iso-pressure contours were very dense, and the non-smoothness at the entrance disappeared in Fig.1(b).

B. Basic Flow

For second-mode disturbance waves:
\[ (u', v', w', p') = A\left[u(y), v(y), w(y), p(y)\right] e^{i\frac{(\omega + \beta \omega - \alpha)}{\Delta x}} \tag{4} \]

For first-mode T-S waves:
\[ (u', v', w', p') = A\left[u(y), v(y), w(y), p(y)\right] e^{i\frac{(\omega + \alpha \omega - \beta)}{\Delta x}} \tag{5} \]

In the computation, \( u(y), u(y), v(y), w(y) \) and \( p(y) \) were the normalized eigen-functions of velocity and pressure of disturbance wave with its frequency \( \omega \) respectively, \( \alpha \) the streamwise wave number, \( \beta \) the spanwise wave number, \( c.c \) denoted complex conjugate. All corresponded eigen-functions were solved from the eigen-value problem of the Orr-Sommerfeld equation, based on the velocity profile at the entrance.

The propagation speed of the wave was 0.91, its wave length was 2.67, and in each wave length, there were 16 meshes. In each period of time, there were 1000 time steps of integration. We have also doubled the number of meshes and time steps respectively, and the results remained almost the same, thus confirming that the mesh size and time step adopted were appropriate.

IV. RESULTS AND DISCUSSION

A. Shocklet Detection Algorithm

We now turn to the existence of shocklets in the present simulation. The problem if there would be shocklets being generated was considered. It was usually determined as the location where the iso-contour of a certain flow variable was very densely populated. However, this method may not be reliable if the shocklets are rather weak.

However, in the regions where shocklets are found the flow in frame relative to a moving normal shock, decelerates from supersonic to subsonic. In present cases, the propagation speed of the shocklets is very close to the propagation speed of the disturbance. So the velocity of a shocklet was very close to 0.91. Upon that the most effective and convincing way to locate a shocklet was to put a row of points a little downstream, follow these particles to check if its velocity was supersonic in front of shock and subsonic behind the shock, while pass through shocklets. A detection algorithm fit for 2-D disturbances[6] was extended to 3-D as follow:

\[
\begin{align*}
\omega - a(t, x - \Delta x, y, z) - \beta \omega(t, x - \Delta x, y, z) < 1 - \varepsilon \\
\frac{\left([\alpha^2 + \beta^2] - a(t, x - \Delta x, y, z)\right)}{\left([\alpha^2 + \beta^2] - a(t, x - \Delta x, y, z)\right)} > 1 + \varepsilon
\end{align*}
\tag{6}
\]

where \( a(t, x, y, z), \omega(t, x, y, z) \) were velocity in \( x, z \) direction, respectively, \( a(t, x, y, z) \) the local sound speed, and \( \varepsilon \) an adjustable parameter. \( \Delta x \) should be chosen in considering the thickness of the computed shocklet. According to the shock capturing capability of the numerical scheme, \( \Delta x \) was chosen to be 2 mesh-sizes, and \( \varepsilon = 0.02 \).

In order to verify if a seemingly "shocklet" was really a shocklet, we check if the entropy of the fluid particle would
have a sudden increase in crossing the shocklets.

**B. Results for Second-mode Waves**

At the entrance, second mode disturbance waves with different amplitudes and spanwise wave numbers(\(\beta\)) were introduced. The \(\beta\) varies in a range of 0.1-1.5. Three of them were discussed in detail and the counting parameters were listed in Table 1. The corresponding eigen-functions for \(\beta=0.1\) were shown in Fig. 2. The initial amplitude \(A=0.03\).

**TABLE I**

<table>
<thead>
<tr>
<th align="left">(\omega)</th>
<th align="left">(\alpha)</th>
<th align="left">(\alpha_i)</th>
<th align="left">(\beta)</th>
</tr>
</thead>
<tbody>
<tr>
<td align="left">Case A</td>
<td align="left">2.12686511</td>
<td align="left">-0.04328233</td>
<td align="left">0.1</td>
</tr>
<tr>
<td align="left">Case B</td>
<td align="left">2.14786511</td>
<td align="left">-0.03929546</td>
<td align="left">0.5</td>
</tr>
<tr>
<td align="left">Case C</td>
<td align="left">2.21186511</td>
<td align="left">-0.02811139</td>
<td align="left">1.0</td>
</tr>
</tbody>
</table>

In Fig.3 we show the amplitudes of different harmonics of case A, 1 denotes the fundamental wave, 2, 3 the second and third harmonic respectively. When the amplitude of fundamental wave has already reached 0.1, the evolution of second and third harmonics started to grow quickly, and strong nonlinearity became dominant. Fig.4 shows the entropy evolution curves following fluid points which might go across shocklets starting from \(x=65\). Near that place, it has an apparent entropy jump indicating that there would be a shocklet. (There were some non-physical oscillations near the jump due to the imperfect of the numerical scheme.)

![Fig. 2 The distribution of a second-mode wave (A)](image)

![Fig. 3 Evolution curves for the amplitudes of disturbance.(A)](image)

![Fig. 4 Evolution curve for entropy, following fluid particle (A).](image)

The time history of entropy of a chosen fluid particle was shown in Fig. 5, three sharp increases are clearly observed. We examine its relative Mach number in Fig.6. Correspondingly, its relative Mach number changed abruptly from supersonic to subsonic also three times at the same time instants of entropy jumps. This demonstrated that there exist shocklets indeed.

![Fig. 5 Time history of the entropy of the particle (A).](image)

![Fig. 6 Time history of relative Mach number of the particle (A).](image)

The results of case B were shown in Fig.7-Fig.10. Similar phenomenon can be seen. As the \(\beta\) increase, the amplified ratio decrease according to linear stability theory. Although amplitude of fundamental is little bit smaller than in case A, there were also three sudden entropy jump coinciding with...
relative Mach number decelerates from supersonic to subsonic three times. But the appearance of shocklets was being postponed and the strength of shocklets weaken.

The increase of the spanwise wave number promoted a dramatically change in case C ($A=0.05$).

![Fig. 8 Evolution curve for entropy, following fluid particles. (B)](image)

![Fig. 9 Time history of the entropy of the particle (B)](image)

![Fig. 10 Time history of relative Mach number of the particle. (B)](image)

The increase of the spanwise wave number promoted a dramatically change in case C ($A=0.05$).

![Fig. 11 Evolution curves for the amplitude of disturbance (C).](image)

Fig. 12 showed entropy evolution curve following fluid particles for which $\beta=1.0$. Curves took upper position represented points more close to the wall. Following the curve in $x$ direction, there were several jumps, but their strength were weaker.

![Fig. 12 Evolution curve for entropy, following fluid particles (C).](image)

![Fig. 13 Time history of the entropy of the particle (C)](image)

![Fig. 14 Time history of relative Mach number of the particle (C).](image)

We check the time history of entropy and relative Mach number of fluid particle in Fig.13 and Fig.14. At the same time instants of entropy jumps, the speed of fluid particle remained supersonic. It revealed clearly that there did not exist shocklets.

There are some comments on our results of simulation. A critical and quite strong assumption of detection algorithm is that the shocklets is normal. The small jumps of entropy along the curve might be responsible for weakly oblique shocklets. But when shocklet's strength is small enough, its effect would be insignificant.

C. Results for First-mode Waves

Recently, the secondary instability of the first-mode primary were investigated at a high supersonic Mach number of 4.5[11, 12]. The question whether the case for first-mode T-S waves
should be analyzed. The parameters for 2-D T-S wave and its subharmonic 3-D waves were shown in Table II. The initial amplitudes are $A_1=0.01$ for 2-D T-S wave, and $A_2=0.02$ for 3-D T-S waves, respectively.

TABLE II

<table>
<thead>
<tr>
<th>Parameter of First-mode Waves</th>
<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\alpha_i$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.35500000</td>
<td>0.434092362</td>
<td>-0.002718454</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.17750000</td>
<td>0.214739695</td>
<td>-0.004898773</td>
<td>1.4111613</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 15 showed the resulting amplitudes of $u$-velocity, there was a quick growth of the fundamental and harmonic disturbances. Contrary to that of second mode disturbances, there is no jump of entropy along the curve in Fig. 16. That means no shocklets would be generated by this first-mode disturbances. Additionally, we surveyed the iso-density contours in Fig. 17 and Fig. 18. There are no densely populated either. That confirmed above conclusion.

![Fig. 15 Evolution curves for the amplitude of disturbance](image)

![Fig. 16 Evolution curve for entropy, following fluid particles](image)

![Fig. 17 Iso-density contours within three period](image)

![Fig. 18 Iso-density contours within four period](image)

V. CONCLUSION

Direct numerical simulation in supersonic boundary layer of a flat plate with Mach number 4.5 have been performed with the aims of investigating the existence of shocklet, which is associated with 3-D disturbances. Our main new results are obtained from a shocklet detection algorithm. It appears to be capable of detecting shocklet and characterizing it in terms of the entropy jump condition.

The results have verified that the spanwise wave number have a remarkable effect on the emergence of shocklet. For $\beta<1$, when the amplitude of the disturbances became large enough, says to exceed 0.1, there did exist shocklets. There were no shocklets for $\beta>1$, no matter what the amplitude was. Otherwise, the nonlinear evolution of first mode 2-D T-S wave and a pair of 3-D T-S waves was examined, no shocklet could be found. The occurrence of shocklet depends on spanwise wave numbers and amplitudes of disturbances. It also depends on types of disturbances. Finally we hope that the present simulation can serve as a guide for the establishment of nonlinear stability theories for supersonic flows.

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REFERENCES

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