Boundary layer flows induced by permeable continuous surfaces stretched with prescribed skin friction

Mohamed Ali and Khaled Al-Salem

Abstract- The boundary layer flow and heat transfer on an isothermal stretched surface moving with prescribed skin friction is studied for permeable surface. Three major cases are studied namely; uniformly moving surface (m = 0), stretched surface with constant skin friction (m = 1/3) and surface moving with constant heat flux (m = 1). Where m is the index of the power law velocity exponent. Similarity solutions are obtained for the boundary layer equations subject to power law temperature and velocity variation. The effect of various governing parameters, such as Prandtl number Pr, suction/injection parameter \( f_w \) and m are studied. The results show that increasing m enhances the dimensionless heat transfer at the suction case while degrade it at the injection case at fixed \( f_w \). However, for fixed m as \( f_w \) increases the dimensionless heat transfer coefficient increases.

Keywords- Stretching surface, Boundary layers, Prescribed skin friction, Suction or injection, similarity solutions.

I. INTRODUCTION

A continuously moving surface through an otherwise quiescent medium has many applications in manufacturing processes. Such processes are hot rolling, wire drawing, spinning of filaments, metal extrusion, crystal Since the pioneer study of Sakiadis [6] who developed a numerical solution for the boundary layer flow field of a stretched surface, many authors have attacked this problem to study the hydrodynamic and thermal boundary layers due to a moving surface [7-16].

Recently, Magyari and Keller [17] have initiated a new research field by solving the induced boundary layer flows of impermeable stretched surfaces using prescribed skin friction boundary condition instead of the usual prescribed velocity boundary condition. Their results have shown substantial deviation of velocity profiles, temperature profiles as well as the wall heat fluxes from that of defined velocity boundary condition when compared on the actual physical scales of the coordinates.

Suction or injection of a stretched surface was introduced by Erickson et al [118] and Fox et al [19] for uniform surface velocity and temperature and by Gupta and Gupta [20] for linearly moving surface. Chen and Char [21] have studied the suction and injection on a linearly moving plate subject to uniform wall temperature and heat flux and the more general case using a power law velocity and temperature distribution at the surface was studied by Ali [22]. Recently, Magyari et al. [23] have reported analytical and computational solutions when the surface moves with rapidly decreasing velocities using the self-similar method, and the flow part of the problem was considered analytically by Magyari and Keller [24] for permeable surface moving with a decreasing velocity for velocity parameters −1/3 and −1/2.

In all papers cited earlier the effect of buoyancy force was neglected and the following papers have taken the buoyancy force into consideration however, suction or injection at the moving surface was relaxed. Such papers are Lin et al [25] for horizontal isothermal plate moving in parallel or reversibly to a free stream. Also the papers by Karwe and Jaluria [4, 5], Kang and Jaluria [26, 27] to obtain the buoyancy effects on moving plate in rolling and extrusion processes, in materials processing, casting process, and in channel flow for thermal processing respectively, and it was found that the effect of thermal buoyancy is more significant when the plate is moving vertically upward than when it is moving horizontally. Ingham [28] studied the existence of the solutions of the boundary layer equations of a uniformly moving vertical plate with temperature inversely proportional to the distance up the plate.
Laminar mixed convection of uniformly moving vertical surface for different temperature boundary conditions are considered by Ali and Al-Yousef [29, 30] and by Ali [31] for stretched surface with rapidly decreasing velocities.

The present paper extends the work of [17] for the surface to be permeable and moving with different power law velocity distribution. However, the analyses are focused on cases of uniformly (m = 0), linearly moving surface (m = 1) and m = 1/3 corresponding to fixed skin friction at the surface for isothermal surface temperature (n = 0) for different Prandtl numbers.

II. MATHEMATICAL ANALYSIS

Consider the steady two-dimensional motions of convective boundary layer flow induced by a moving surface with suction or injection at the surface. For incompressible viscous fluid environment with constant properties, the equations governing this convective flow can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(1)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}
\]

(2)

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\alpha}{\nu} \frac{\partial^2 T}{\partial y^2}
\]

(3)

subject to the following prescribed skin friction coefficient boundary conditions:

\[
\tau(x) = \tau_w(x), \quad v(x) = v_w(x) \quad @ y = 0
\]

(4)

\[
T - T_\infty = T_w - T_\infty = Cx^n \quad @ y = 0
\]

\[
u(x) = 0, T(x) = T_\infty = \text{const.} \quad @ y \to \infty
\]

The x coordinate is measured along the moving surface from the point where the surface originates, and the y coordinate is measured normal to it (Fig. 1). Positive or negative v imply injection or suction at the surface respectively, and u and v are the velocity components in x and y directions respectively. Similarity solutions arise when

\[
u = U_0 x^m f' (\eta), \quad T - T_\infty = Cx^n \theta (\eta)
\]

(5)

\[
\eta = y \sqrt{\frac{m+1}{2}} \left( \frac{U_0 x^m}{v x} \right) = \frac{y}{x} \sqrt{\frac{m+1}{2}} \sqrt{Re_x}
\]

(6)

\[
\nu = \frac{2vU_0}{m+1} x^{\frac{m+1}{2}} \left( \frac{m+1}{2} f + \frac{m-1}{2} f' \eta \right), \quad m \neq -1
\]

(7)

and the shear stress and the heat flux at the surface are given by respectively

\[
\tau_w(x) = \mu U_0 x^{m-1} \frac{m+1}{2} Re_x^{1/2} f^*(0) \sim x^{(5m-3)/2} f^*(0)
\]

(8)

\[
q_w(x) = -kC x^{n-1} Re_x^{1/2} \left( \frac{m+1}{2} \right)^{1/2} \theta(0) \sim x^{(m+12n)/2} \theta'(0)
\]

(9)

It should be mentioned that, positive or negative m indicate that the surface is accelerated or decelerated from the extruded slit respectively. Where f' and \( \theta' \) are the dimensionless velocity and temperature respectively, and \( \eta \) is the similarity variable. Substitution in the governing equations gives rise to the following two-point boundary-value problem.

\[
f'' + f = - \frac{2m}{m+1} f'^2 = 0
\]

(10)

\[
\theta'' + Pr \left( f\theta' - \frac{2n}{m+1} f' \theta \right) = 0
\]

(11)

The transformed boundary conditions for prescribed skin friction coefficient following Magyari and Keller [17].

\[
f^*(0) = -1, \quad f(0) = f_w, \quad f'(\infty) = 0
\]

(12)

\[
\theta(0) = 1, \quad \theta(\infty) \to 0
\]

(13)

The quantity \( f(0) = f_w \) will be referred to as the dimensionless suction/injection velocity. In this way \( f_w = 0 \) corresponds to an impermeable surface, \( f_w < 0 \) to suction (i. e. \( v_w(x) < 0 \)) and \( f_w > 0 \) to lateral injection (i. e. \( v_w(x) > 0 \)) of the fluid through a permeable surface. The temperature of the injected fluid is assumed to coincide with the local temperature \( T_w(x) \) of the stretching surface.

The local skin friction coefficient and the local Reynolds and Nusselt numbers are now given at the surface by

\[
C_t = \frac{\sqrt{Re_x}}{2} = f^*(0) \sqrt{\frac{m+1}{2}}
\]

(14)

\[
Re_x = \frac{U_w x^{m+1}}{v}
\]

(15)
\[ \frac{\text{Nu}}{\sqrt{\text{Re} \cdot \eta}} = -\sqrt{m+1} \quad \theta'(0) \quad (16) \]

III. NUMERICAL SOLUTION PROCEDURE

The coupled nonlinear ordinary differential equations (10) and (11) are solved numerically by using the fourth order Runge-Kutta method. Solutions of the differential Eqs. (10) and (11) subject to the boundary conditions (12), (13) were solved for different values of \( f_w \). At each \( f_w \), guessed values for \( \dot{f}'(0) \) and \( \theta'(0) \) are assumed and the differential equations (10) and (11) were integrated until the boundary conditions at infinity \( \dot{f}'(\eta) \) and \( \theta(\eta) \) decay exponentially to zero (at least of order \( 10^{-5} \)). If the boundary conditions at infinity are not satisfied then the numerical routine uses a half interval method to calculate corrections to the estimated values of \( \dot{f}'(0) \) and \( \theta'(0) \). This process is repeated iteratively until exponentially decaying solution in \( \dot{f}' \) and \( \theta \) is obtained. The value of \( \eta_{\infty} \) was chosen as large as possible depending upon the Prandtl number and the suction/injection parameter \( f_w \), without causing numerical oscillations in the values of \( \dot{f}' \) and \( \theta \). It should be noted that since the \( \dot{f}' \) and \( \theta \) must exponentially decaying to zero to satisfying the boundary conditions at infinity then implied boundary conditions which must also be satisfied are \( \dot{f}'(\eta) \) and \( \theta'(\eta) \) approaching zeros at infinity. Numerical solutions are obtained for a range of \( f_w \) between a minimum negative value (injection) to a maximum positive value (suction) such that the boundary conditions mentioned earlier must satisfy otherwise the solutions are rejected. Comparison with Magyari and Killer [17] is given in Table 1 for uniform skin friction (\( m = 1/3 \)) and for uniform surface temperature (\( n = 0 \)) and for \( Pr = 100 \). This comparisons show good agreements in \( \dot{f}'(0) \), \( f(\infty) \) and \( \theta'(0) \) which gives confidence about our current results.

Table 1. Comparisons with Magyari and Killer [17] for isothermal impermeable surface stretched with constant skin friction (\( n = 0 \), \( m = 1/3 \)) in an ambient fluid medium of \( Pr = 100 \)

<table>
<thead>
<tr>
<th>( f_w )</th>
<th>( \dot{f}'(0) )</th>
<th>( \theta'(0) )</th>
<th>( f(\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.1323120</td>
<td>-8.300242</td>
<td>1.130436</td>
</tr>
</tbody>
</table>

Magyari and Killer [17]

<table>
<thead>
<tr>
<th>( f_w )</th>
<th>( \dot{f}'(0) )</th>
<th>( \theta'(0) )</th>
<th>( f(\infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.1323139</td>
<td>-8.3001</td>
<td>1.13030744</td>
</tr>
</tbody>
</table>

IV. RESULTS AND DISCUSSION

Equations (10) and (11) are solved numerically for uniformly moving surface velocity with uniform temperature (\( m = 0 \), \( n = 0 \)), for constant shear stress at the surface and isothermal temperature (\( m = 1/3 \), \( n = 0 \)) and for uniform heat flux at the surface (\( m = 1 \), \( n = 0 \)) for \( Pr = 0.72 \), 3 and 7. Figure 2(a, b) shows the velocity profiles in 2(a) and the temperature profiles of both velocity and temperature. Figure 4(a, b) shows the corresponding profiles for uniform shear stress (\( m = 1/3 \)) at the surface with isothermal surface temperature. The effect of increasing \( m \) is clear on the profiles of both velocity and temperature. Figure 4(a, b) shows the velocity and temperature profiles for \( m = 1 \) and \( n = 0 \) corresponding to uniform heat flux at the surface as defined.
Figure 3. Profiles induced by stretched surface with constant skin friction for Pr = 0.72, m = 1/3 and n = 0 (a) velocity and (b) temperature.

by Eq. (9). Comparison between Figs. 2, 3 and 4 shows that the velocity profiles are function of m as expected and the temperature profiles are function of both m and n however, significant increase in the thickness of the thermal boundary layer occur for injection but no major changes happen in the thickness in case of suction. Figure 5 shows the velocity profiles for different m as a function of $f_w$. This figure shows that as m increases the velocity at the surface decreases at fixed $f_w$ up to about $f_w = 2$ then no changes could be observed for changing m. The vertical dashed line presents the impermeable case.

Figure 4. Profiles induced by stretched surface with constant skin friction for Pr = 0.72, m = 1 and n = 0 (a) velocity and (b) temperature

Figure 5. Profiles induced by stretched surface with constant skin friction for various values of m (a) stretching velocity and (b) entrainment velocity

Figure 6. Dimensionless heat transfer coefficient as defined by Eq. (16) for various values of m, Pr = 0.72, uniform surface temperature (n = 0) as function of the suction/injection parameter $f_w$.

The dimensionless heat transfer coefficient is presented in Fig. 6 for Pr = 0.72 and for different values of m. It could be seen from this figure that for isothermal surface temperature increasing m enhances the heat transfer at the suction case while degrades it at the injection case at any fixed $f_w$. 
However, for fixed $m$ as $f_w$ increases the dimensionless heat transfer coefficient increases too.

![Figure 7. Dimensionless heat transfer coefficient as defined by Eq. (16) for various values of Prandtl numbers and for uniform surface temperature and for uniformly moving surface velocity ($m = 0$) as function of the suction/injection parameter $f_w$.](image)

It should be mentioned that other Prandtl numbers give similar results and comparisons of the dimensionless heat transfer coefficient for different Prandtl numbers is shown in Fig. 7 for the uniform moving surface with uniform surface temperature. As expected increasing Prandtl number enhances the heat transfer coefficient especially in the suction case.

V. CONCLUSIONS

Heat transfer and flow field characteristics of a isothermal stretched permeable surface with prescribed skin friction are studied for uniformly moving surface, surface stretched with constant skin friction and for linearly moving surface.

The results show that at the surface the dimensionless stretching velocity decreases as the velocity index increases for any fixed suction/injection parameter $f_w$. It is also shown that the effect of increasing $m$ is negligible beyond $f_w \geq 2.0$. For isothermal surface temperature increasing $m$ enhances the dimensionless heat transfer (Eq. (16)) at the suction side and degrades it at the injection side at fixed $f_w$. However, for fixed $m$ as $f_w$ increases the dimensionless heat transfer coefficient increases too. Furthermore, the dimensionless heat transfer coefficient is function of Prandtl number and enhancements are occurred as the Pr increases for fixed values of $f_w$. This enhancement almost obtained in the suction region while almost no remarkable distinction at the injection side as $f_w$ approaches -2.

ACKNOWLEDGMENT

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through the research group project No RGP-VPP-080.

REFERENCES


Mohamed E. Ali: Was born in Alexandria, Egypt. He obtained his BS (1978) in Mechanical Power Engineering from Helwan University, Cairo Egypt then he obtained his MS (1984) in Mechanical Engineering (thermal sciences option) from University of Colorado at Boulder, USA where he obtained his PhD (1988) in the same major and minor specification from the same university.

He worked in King Saud University at Mechanical Engineering Department, Riyadh 11421, Saudi Arabia as a Professor of Heat Transfer and Thermodynamics. His main research interests are the stability of fluids with heat transfer, numerical and semi-analytical heat transfer of stretched surface, experimental and numerical convection heat transfer, nano-fluid heat transfer. He has published more than 65 articles in well-recognized journals and proceedings.

Prof. Ali is a referee for most international journals in his field. He has collaborated in research with professors at university of Colorado at Boulder, Northwestern University at Evanston, IL, and Swiss Federal Institute of Technology, Zürich, Switzerland. He is an Editor-in-Chief of Journal of Experimental and numerical Heat Transfer (ENHT) and is an Editorial Board member of ISRN Mechanical Engineering and Editorial Board of the Fundamental Journal of Thermal Science and Engineering.

Khaled Al-Salem is an assistant professor at the department of mechanical engineering, King Saud University (KSU), Saudi Arabia. He has earned his Bs degree in mechanical engineering from KSU in 1998, then he proceeded to continue his graduate studies at the University of New York at Buffalo in the field of Fluid Dynamics, aero acoustics and the related field of Heat Transfer. During the course of his graduate studies he earned an advanced certificate of computational science from the Center ofComputational Science Research. In 2005, he earned his PhD in mechanical engineering focusing on turbulent modeling and computational fluid dynamics in general. Then he went back to KSU and was appointed as an assistant professor at the department of mechanical engineering and has been working on several research projects, from which he published several research papers covering subjects in fluid mechanics and heat transfer. Now, he is serving as the vice dean of the college of engineering at KSU.