Controllability of nonlinear systems

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Abstract: In the present paper local constrained controllability problems for nonlinear and semilinear system with constant delays in the control are formulated and discussed. Using some mapping theorems taken from functional analysis and linear approximation methods sufficient conditions for relative and absolute local constrained controllability in a given time interval are derived and proved. The present paper extends controllability conditions with unconstrained controls given in the literature to cover the case of nonlinear systems with delays in control and with constrained controls. Special case of semilinear systems without delays in control is also discussed. As an illustrative example constrained local controllability of semilinear mathematical model of antiangiogenic therapy is considered.

Key-Words: Controllability. Nonlinear systems. Systems with delays in control. Semilinear control systems.

1. Introduction
Controllability is one of the fundamental concept in mathematical control theory. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability which depend on a class of dynamical system [1-8].

Up to the present time the problem of controllability in continuous and discrete time linear dynamical systems has been extensively investigated in many papers (see e.g., [1], [3], or [5] for the extensive list of publications). However, this is not true for the nonlinear dynamical systems specially with delays in control and with constrained controls. Only a few papers concern constrained controllability problems for continuous or discrete nonlinear dynamical systems.

In the present paper local relative and absolute constrained controllability problems for nonlinear system with many constant delays in control are formulated and discussed. Using some mapping theorems taken from functional analysis [2], [5] and linear approximation methods sufficient conditions for constrained relative and absolute controllability are derived and proved. Special case of semilinear systems without delays in control is also considered.

The present paper extends in some sense the results given in papers [2], [4] and [5] to cover the nonlinear systems with delays in control and with constrained controls.

2. Preliminaries
Let us consider general nonlinear system with constant delays described by the following differential equation

\[ x'(t) = f(x(t),u(t-h_0),u(t-h_1),...,u(t-h_i),...,u(t-h_M)) \] (1)

where:

\[ 0=h_0<h_1<...<h_i<...<h_M \]

are constant delays,

\[ x(t)\in\mathbb{R}^n \] is a state vector at the time \( t \),

\[ u(t)\in\mathbb{R}^m \] is a control vector at the time \( t \),
f: \( R^n \times R^m \times R^m \times \cdots \times R^m \times \cdots \times R^m \to R^n \),

is a given function, which is continuously differentiable with respect all its arguments in some neighborhood of zero and

\[ f(0,0,\ldots,0) = 0. \]

Let \( U \subset R^n \) be a given arbitrary set. Let \( L_\infty([0,t_1],U) \) be the set of admissible controls. In the sequel we shall also use the following notations: \( \Omega^0 \subset R^m \) is a neighborhood of zero, \( U^c \subset R^m \) is a closed convex cone with vertex at zero and \( U^c = U^c \cap \Omega^0 \). Moreover, let us observe, that for \( u=U^c \) the set of admissible controls \( L_\infty([0,t_1],U) \) is a cone in the linear space \( L_\infty([0,t_1],R^m) \).

The initial conditions for nonlinear differential equation (1) are given by

\[ x(0) = x_0 \in R^n, \quad u_0 \in L_\infty([-h_M,0],U) \] (2)

where \( x_0 \) is a known vector and \( u_0 \) is a known function.

For a given initial conditions (2) and for an arbitrary admissible control \( u \in L_\infty([0,t_1],U) \) there exists unique solution of the nonlinear differential equation (1) for \( t \in [0,t_1] \).

Since function \( f \) is continuously differentiable therefore, using the standard methods, it is possible to construct linear approximation of the nonlinear system (1). This linear approximation is valid in some neighborhood of the point zero in the product space \( R^n \times R^m \times \cdots \times R^m \times \cdots \times R^m \), and is given by the linear differential equation (3)

\[ x'(t) = Ax(t) + \sum_{i=0}^{i=M} B_i u(t - h_i) \] (3)

defined for \( t \geq 0 \), where \( A \) is \( n \times n \)-dimensional constant matrix, and \( B_i, i=0,1,2,\ldots,M \) are \( n \times m \)-dimensional constant matrices given by:

\[ A = f'_x(0,0,0,\ldots,0) \]

\[ B_i = f'_u(0,0,0,\ldots,0), \quad \text{for} \quad i=0,1,2,\ldots,M \]

For linear system (3) we can define the so called state transition matrix \( \exp(At) \). Using the state transition matrix \( \exp(At) \) we can express the solution \( x(t;x_0,u_0,u) \) of linear system (2.3) for \( t>h_M \) in the following compact form

\[ x(t;x_0,u_0,u) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau)) \left( \sum_{i=0}^{i=M} B_i u(\tau - h_i) \right) d\tau = \]

\[ \exp(At)x_0 + \sum_{i=0}^{i=M} \int_0^t \exp(A(t+s+h_i))B_i u_0(s)ds + \sum_{i=0}^{i=M} \int_0^t \exp(A(t+s+h_i))B_i u(s)ds \] (4)

For zero initial condition

\[ x(0)=x_0=0, \quad u_0=0, \]

the solution \( x(t;0,0,u) \) to (3) for \( t>h_M \) is given by

\[ x(t;0,0,u) = \sum_{i=0}^{i=M} \int_0^t \exp(A(t+s+h_i))B_i u(s)ds \] (5)

Finally, let us also define the associated linear system without delays

\[ x'(t) = Ax(t) + Du(t) \] (6)

where

\[ D = \sum_{i=0}^{i=M} \exp(Ah_i)B_i \]

For linear and nonlinear systems with delays in control it is possible to define many different concepts of controllability [2], [3], [4]. In the sequel, we shall concentrate on local and global relative and absolute \( U \)-controllability in a given time interval \([0,t_1] \), where \( t_1 > h_M \).

Definition 2.1. System (1) is said to be globally relatively \( U \)-controllable in a given interval \([0,t_1] \), if for zero initial conditions \( x_0=0, \quad u_0=0 \) and every vector \( x' \in R^n \), there exists an admissible control \( u \in L_\infty([0,t_1],U) \), such that the corresponding solution of the equation (1) satisfies condition \( x(t_1;0,0,u) = x' \).
Definition 2.2. System (1) is said to be locally relatively $U$-controllable in a given interval $[0,t_1]$ if for zero initial conditions $x_0=0$, $u_0=0$, there exists neighborhood of zero $D_x\subset\mathbb{R}^n$, such that for every point $x\in D_x$ there exists an admissible control $u\in L_{x,\ell}([0,t_1], U)$, such that the corresponding solution of the equation (1) satisfies the condition $x(t_1;0,0,u)=x'$.

Definition 2.3. System (1) is said to be globally absolutely $U$-controllable in a given interval $[0,t_1]$, if for zero initial conditions $x_0=0$, $u_0=0$, any given function $u_1\in L_{x,\ell}([t_1-h_d,t_1], U)$, and every vector $x'\in\mathbb{R}^n$, there exists an admissible control $u\in L_{x,\ell}([0,t_1-h_m], U)$, such that the corresponding solution of the equation (1) satisfies condition $x(t_1;0,0,u)=x'$.

Definition 2.4. System (1) is said to be locally absolutely $U$-controllable in a given interval $[0,t_1]$ if for zero initial conditions $x_0=0$, $u_0=0$, any given function $u_1\in L_{x,\ell}([t_1-h_d,t_1], U)$ there exists neighborhood of zero $D_x\subset\mathbb{R}^n$, such that for every point $x\in D_x$ there exists an admissible control $u\in L_{x,\ell}([0,t_1-h_m], U)$, such that the corresponding solution of the equation (1) satisfies condition $x(t_1;0,0,u)=x'$.

Of course the same definitions are valid for linear systems (3). For linear systems with delays in control various controllability conditions are presented in the literature (see e.g., [1], [3] or [4]). It is well known [2], that for the sets $U$ containing zero as an interior point, the local relative (absolute) constrained controllability is equivalent to the global relative (absolute) unconstrained controllability.

Lemma 2.1. [2] Linear system (3) is locally relative (absolute) $\Omega$-controllable in the time interval $[0,t_1]$ if and only if it is globally relative (absolute) $\mathbb{R}^n$-controllable in the time interval $[0,t_1]$.

Corollary 2.1 directly follows from the well known fact, that the range of the linear bounded operator covers whole space if and only if this operator transforms some neighborhood of zero onto some neighborhood of zero in the range space [1].

Finally, it should be pointed out, that absolute local (global) constrained controllability of the linear system with delays (3) in the time interval $[0,t_1]$ is equivalent to local (global) constrained controllability in the time interval $[0,t_1-h_m]$ of the associated linear system without delays (6).

Lemma 2.2. [4] Linear system (3) is locally (globally) absolute $U$-controllable in the interval $[0,t_1]$ if and only if the associated linear system without delays is locally (globally) $U$-controllable in the interval $[0,t_1-h_m]$.

### 3. Relative controllability

In this section we shall formulate and prove sufficient conditions of the local $U$-controllability in a given interval $[0,t_1]$ with different sets $U$ for the nonlinear system (1). Proofs of the main results are based on some lemmas taken directly from functional analysis and concerning so called nonlinear covering operators [1], [5]. Now, for convenience we shall shortly state this results.

Lemma 2.1. [2], [5] Let $F: Z\rightarrow Y$ be a nonlinear operator from a Banach space $Z$ into a Banach space $Y$ and suppose that $F(0)=0$. Assume that the Frechet derivative $dF(0)$ maps a closed convex cone $C\subset Z$ with vertex at zero onto the whole space $Y$. Then there exists neighborhoods $M_0\subset Z$ about $0\in Z$ and $N_0\subset Y$ about $0\in Y$ such that the equation $y=F(z)$ has for each $y\in N_0$ at least one solution $z\in M_0\cap C$.

Let us observe, that a direct consequence of Lemma 3.1 is the following result concerning nonlinear covering operators.

Lemma 2.2. [2], [5] Let $F: Z\rightarrow Y$ be a nonlinear operator from a Banach space $Z$ into a Banach space $Y$ which has the Frechet derivative $dF(0): Z\rightarrow Y$, whose image coincides with the whole space $Y$. Then the image of the operator $F$ will contain a neighborhood of the point $F(0)\in Y$.

Now, we are in the position to formulate and prove the main result on the local relative $U$-controllability in the interval $[0,t_1]$ for the nonlinear system (1).

Theorem 2.1. Let us suppose, that $U^c\subset \mathbb{R}^m$ is a closed convex cone with vertex at zero. Then the nonlinear system (1) is locally relatively $U$-controllable in the interval $[0,t_1]$ if its linear approximation near the origin given by the differential equation (3) is globally relatively $U^c$-controllable in the same interval $[0,t_1]$.

Proof. Proof of the Theorem 2.1 is based on Lemma 2.1. Let our nonlinear operator $F$ transforms the space of admissible controls $L_{x,\ell}([0,t_1], U^c)$ into the space $\mathbb{R}^n$ at the time $t_1$ for the nonlinear system (1). More precisely, the
nonlinear operator $F: \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined as follows

$$F(L_c([0,t_1],U^c)) = x(t_1,0,0,u) \quad (7)$$

where $x(t_1,0,0,u)$ is the solution at time $t_1 > h_M$ of the nonlinear system (1) corresponding to an admissible control $u \in L_c([0,t_1],U^c)$ and for zero initial conditions. Frechet derivative at point zero of the nonlinear operator $F$ denoted as $dF(0)$ is a linear bounded operator defined by the following formula

$$dF(0)(L_c([0,t_1],U^c)) = x(t_1,0,0,u) \quad (8)$$

where $x(t_1,0,0,u)$ is the solution at time $t_1$ of the linear system (3) corresponding to an admissible control $u \in L_c([0,t_1],U^c)$ and for zero initial conditions.

Since $f(0,0,...,0,...)=0$, then for zero initial conditions nonlinear operator $F$ transforms zero into zero i.e., $F(0)=0$. If linear system (3) is globally relatively $U^c$-controllable in the interval $[0,t_1]$, then the image of Frechet derivative $dF(0)$ covers whole space $\mathbb{R}^n$. Therefore, by the result stated at the beginning of the proof, the nonlinear operator $F$ covers some neighborhood of zero in the space $\mathbb{R}^n$. Hence, by Definition 2.2 nonlinear system (1) is locally relatively $U^c$-controllable in the interval $[0,t_1]$.

In the case when the set $U$ contains zero as an interior point, then by Lemma 3.2 we have the following sufficient condition for local constrained relative controllability of nonlinear system (1).

Corollary 2.1. Let $0 \in \text{int}(U)$. Then the nonlinear system (3.1) is locally relatively $U$-controllable in the interval $[0,t_1]$ if its linear approximation near the origin given by the differential equation (3) is globally relatively $\mathbb{R}^n$-controllable in the same interval $[0,t_1]$.

4. Absolute controllability

Using similar methods as in Section 3 it is possible to derive sufficient conditions for local absolute $U^c$-controllability in a given interval $[0,t_1]$, $t_1 > h_M$ for the nonlinear system (2.1).

Theorem 3.1. Let us suppose, that $U^c \subset \mathbb{R}^n$ is a closed convex cone with vertex at zero. Then the nonlinear system (1) is locally absolutely $U^c$-controllable in the interval $[0,t_1]$ if its linear approximation near the origin given by the differential equation (3) is globally absolutely $U^c$-controllable in the same interval $[0,t_1]$.

Corollary 4.1. Let $0 \in \text{int}(U)$. Then the nonlinear system (7) is locally absolutely $U$-controllable in the interval $[0,t_1]$ if its linear approximation near the origin given by the differential equation (3) is globally absolutely $\mathbb{R}^n$-controllable in the same interval $[0,t_1]$.

5. Semilinear systems

Semilinear dynamical systems are special and important cases of general nonlinear dynamical systems. In this section using the results of the previous sections, we study special case of general nonlinear system (1) with
delays in controls namely, the semilinear stationary finite-dimensional control system without delays in control described by the following ordinary differential state equation

\[ x'(t) = Ax(t) + F(x(t),u(t)) + Bu(t) \quad \text{for } t \in [0,t_1], \quad (11) \]

with zero initial conditions: \( x(0) = 0 \)

where the state \( x(t) \in \mathbb{R}^n \)

the control \( u(t) \in \mathbb{R}^m \),

\( A \) is \( n \times n \) dimensional constant matrix,

\( B \) is \( n \times m \) dimensional constant matrix.

Moreover, let us assume that the nonlinear mapping

\[ F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \]

is continuously differentiable near the origin and such that \( F(0,0)=0 \).

\[ z'(t) = Cz(t) + Du(t) \quad \text{for } t \in [0,t_1] \quad (12) \]

with zero initial condition \( z(0)=0 \), where

\[ C = A + D_x F(0,0) \quad D = B + D_u F(0,0) \quad (13) \]

are \( n \times n \)-dimensional and \( n \times m \)-dimensional constant matrices, respectively.

The main result of this section is the following sufficient condition for constrained local controllability of the semilinear dynamical system (11).

Theorem 5.1 Suppose that

(i) \( F(0,0) = 0 \),

(ii) \( U_c \subset \mathbb{R}^m \) is a closed and convex cone with vertex at zero,

(iii) The associated linear control system (31) is \( U_c \)-globally controllable in \([0,t_1]\).

Then the semilinear stationary dynamical control system (1) is \( U_c \)-locally controllable in \([0,t_1]\).

It should be pointed out, that in applications of the Theorem 5.1, the most difficult problem is to verify the assumption (iii) about constrained global controllability of the linear stationary dynamical (12). In order to avoid this disadvantage, we may use the following well known Theorem 5.2.

Theorem 5.2 [1], [2]. Suppose the set \( U_c \) is a cone with vertex at zero and nonempty interior in the space \( \mathbb{R}^m \).

Then the associated linear dynamical control system (12) is \( U_c \)-globally controllable in \([0,t_1]\) if and only if

(1) it is controllable without any constraints, i.e.

\[ \text{rank}[D,CD,C^2D,...,C^{n-1}D] = n, \]

(2) there is no real eigenvector \( v \in \mathbb{R}^n \) of the matrix \( C^m \)

satisfying inequalities

\[ v^T D u \leq 0, \text{ for all } u \in U_c. \]

It should be pointed out that for the special case namely, for the single input scalar control constrained controllability conditions for linear system (12) are rather simple. In this case i.e., for the case \( m=1 \), Theorem 5.2 reduces to the following Corollary.

Corollary 5.1. [1],[2]. Suppose that \( m=1 \) and \( U_c=\mathbb{R}^+ \).

Then the associated linear dynamical control system (31) is \( U_c \)-globally controllable in \([0,t_1]\) if and only if it is controllable without any constraints i.e.,

\[ \text{rank}[D,CD,C^2D,...,C^{n-1}D] = n, \]

and matrix \( C \) has only complex eigenvalues.

6. Example

In this section as an illustrative example we shall consider constrained local controllability of semilinear mathematical model of antiangiogenic therapy, described by the following semilinear differential state equations without delays:

\[ x'(t) = y - x, \]

\[ y' = -\partial(e^{2/3x} - 1) + \sigma u, \]

Therefore, taken into account the general form of semilinear dynamical systems we have

\[ A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \]
\[ F(x,u) = F(x) = \begin{bmatrix} 0 \\ -\vartheta e^{x/3} - 1 \end{bmatrix} \]
\[ B = \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \]

Hence, we have
\[ F_x(0,0) = \begin{bmatrix} 0 & 0 \\ -\vartheta^2 & 0 \end{bmatrix} \]
\[ C = A + F_x(0,0) = \begin{bmatrix} -1 & 1 \\ -\vartheta & 0 \end{bmatrix} \]
\[ \text{rank}[CB] = \text{rank} \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix} = 2 = n \]

Moreover,
\[ \det(sI - C) = \det \begin{bmatrix} s + 1 & -1 \\ \vartheta & s \end{bmatrix} = (s + 1) + \frac{2}{3} \vartheta = s^2 + s + \frac{2}{3} \vartheta \]

Hence, \( \Delta = 1 - \frac{8}{3} \vartheta \) and for \( \vartheta > \frac{3}{8} \) and matrix C has only complex eigenvalues, and system is constrained locally controllable.

### 7. Conclusion

In the present paper different types of controllability of nonlinear control system with constant delays in control has been considered. The results presented can be extended in many directions. For example it is possible to formulate sufficient local controllability conditions for nonlinear time-varying systems. Moreover, similar controllability results can be derived for very general nonlinear systems with distributed and several time-dependent delays in the control and for nonlinear infinite-dimensional systems with different kinds of delays. As an illustrative example constrained local controllability of semilinear mathematical model of antiangiogenic therapy, described by the semilinear differential state equations without delays was considered.

### References


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