Some Congruences of Fibonacci and Lucas numbers
and Properties of Fibonacci Functions

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Abstract: In this paper we prove some congruences of Fibonacci and Lucas numbers. We also prove that the Box
dimension of the Fibonacci functions defined on [0, 1] is less than 1.

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MSC 2000: 11D25, 11B39

1 Introduction
Let \( (F_n)_{n \geq 0} \),

\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0,
\]

be the Fibonacci sequence and \( (L_n)_{n \geq 0} \)

\[
L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0,
\]

be the Lucas sequence.

It is known that

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right],
\]

\[
L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

Let

\[
f(x) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^x - \left( \frac{1 - \sqrt{5}}{2} \right)^x \cos(\pi x) \right), \quad x \in \mathbb{R}
\]

be the Fibonacci function.

Let \( p \) be a prime odd integer. The following result

about the Fibonacci and Lucas numbers is known:

**Theorem 1.1 (Legendre, Lagrange).** Let \( p \) be a prime odd integer. Then the Fibonacci number, \( F_p \), has the property:

\[
F_p \equiv \left( \frac{p}{5} \right) (\text{mod } p)
\]

and the Lucas number, \( L_p \), satisfies:

\[
L_p \equiv 1 (\text{mod } p).
\]
We remember the following theorems that will be used to prove the results in the next chapter:

**Fermat’s little theorem.** If \( p \) is a prime number, then for any integer \( a \), \( a^p - a \) is evenly divisible by \( p \).

**Remark.** A variant of the theorem is the following: If \( p \) is a prime and \( a \) is an integer coprime to \( p \), then \( a^{p-1} \equiv 1 \pmod{p} \).

**Euler’s criterion.** Let \( p \) be an odd prime integer and \( a \) an integer number coprime with \( p \). Then
\[
\left( \frac{a}{p} \right) \equiv a^{(p-1)/2} \pmod{p}.
\]

**The quadratic reciprocity law.** Let \( p \) and \( q \) be two distinct odd prime integers. Then
\[
\left( \frac{p}{q} \right) \cdot \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.
\]

In this paper we find some congruences satisfied by the Fibonacci numbers \( F_{2p} \) and Lucas numbers \( L_{2p} \), where \( p \) is a prime odd integer.

In our proof we use techniques of combinatorics or elementary number theory.

Let \( F \) be a nonempty and bounded subset of \( \mathbb{R}^2 \), \( N_\delta(F) \) the least number of sets whose union covers \( F \) with and diameters that do not exceed a given \( \delta > 0 \). Then, the upper bound Box dimension of \( F \) is defined by:
\[
\dim_F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{- \log \delta}.
\]

Let \( f : I \to \mathbb{R} \) be a function defined on the interval \( I \) and \( [t_1, t_2] \) be a subinterval of \( I \). We denote by:
\[
R_f(t_1, t_2) = \sup_{t_1 \leq u \leq t_2} |f(t) - f(u)|,
\]
and by \( \Gamma(f) \) the graph of \( f \).

**Lemma 1.1.** Let \( f \) be a continuous function on \( [0, 1] \), \( 0 < \delta < 1 \), and \( m \) be the least integer greater than or equal to \( 1/\delta \). If \( N_\delta \) is the number of the squares of the \( \delta \)-mesh that intersect \( \Gamma(f) \). Then:
\[
\delta^{-1} \sum_{j=0}^{m-1} R_f[j\delta, (j+1)\delta] \leq N_\delta \leq 2m + \delta^{-1} \sum_{j=0}^{m-1} R_f[j\delta, (j+1)\delta].
\]

**2 Congruences of Fibonacci and Lucas numbers**

**Lemma 2.1** Let \( n \) be a positive integer, \( n \geq 2 \) and \( p \) a prime odd integer. Then \( p \) divides \( \binom{np}{p} - n \).

**Proof.**
The proof is simple, using the congruence
\[
\left( \frac{pa}{pb} \right) \equiv \left( \frac{a}{b} \right) \pmod{p},
\]
where \( a \) and \( b \) are positive integers, \( b \leq a \) and \( p \) is a prime integer or using induction after \( n \in \mathbb{N}^* \) and the identity
\[
\sum_{k=0}^{m} \binom{n}{k} \binom{t}{m-k} = \binom{n+t}{m}.
\]

**Lemma 2.2** Let \( p \) be a prime odd integer. Then \( p \) divides \( \binom{2p}{k} \), for \( 1 \leq k \leq 2p - 1 \), \( k \neq p \).

**Proof.**
\[
\binom{2p}{k} = \frac{(2p)!}{k!(2p-k)!} = \frac{1 \cdot 2 \cdot \ldots \cdot p \cdot \ldots \cdot (2p)}{k!(2p-k)!}.
\]
If \( 1 \leq k \leq p - 1 \), it results that \( p \) doesn’t divide \( k \) and \( p + 1 \leq 2p - k \leq 2p - 1 \).

This implies that:
\[
p|(2p - k)! \text{ and } p^2 \text{ doesn’t divide }(2p - k)!
\]
Thus:
\[
\binom{2p}{k}.
\]

If \( p + 1 \leq k \leq 2p - 1 \), it results that
\[
p^k! \text{ and } p^2 \text{ doesn’t divide } k!.
\]
We have \( 1 \leq 2p - k \leq p - 1 \), so \( p \) doesn’t divide \( (2p - k)! \).
Therefore, \( p \left( \left\lfloor \frac{2p}{k} \right\rfloor \right) \).

**Proposition 2.3** Let \( p \) be a prime odd integer. Then the Lucas number \( L_{2p} \equiv 3 \pmod{p} \).

**Proof.**

\[
L_{2p} = \left( \frac{1 + \sqrt{5}}{2} \right)^{2p} + \left( \frac{1 - \sqrt{5}}{2} \right)^{2p} = \frac{1}{2^{2p-1}} \sum_{k=0}^{2p} \binom{2p}{k} \cdot \frac{k}{5^2}.
\]

Using Lemma 2.2, we obtain that

\[
L_{2p} \equiv \frac{1}{2^{2p-1}} \cdot (1 + 5^p) \pmod{p} \iff L_{2p} \equiv \frac{2}{4^p} \cdot (1 + 5^p) \pmod{p}.
\]

Applying Fermat’s small theorem we obtain:

\[ L_{2p} \equiv 3 \pmod{p}. \]

**Corollary 2.5** Let \( p \) be a prime odd integer. Then:

\[ p \left( \left\lfloor \frac{2p}{k} \right\rfloor \right) \pmod{p}. \]

**Proof.** It is know that between Fibonacci and Lucas numbers there is the following connection:

\[ L_n = F_{n+1} + F_{n-1}. \]

So,

\[ L_{2p} = F_{2p+1} + F_{2p-1}. \]

Applying Proposition 2.3, Proposition 2.4 and the properties of Fibonacci and Lucas numbers, we obtain:

\[ F_{2p} = \left( \frac{5}{p} \right) \pmod{p}. \]

Applying the quadratic reciprocity law, we have:

\[ \left( \frac{p}{5} \right) \cdot \left( \frac{5}{p} \right) = (-1)^{p-1} = 1 \iff \left( \frac{5}{p} \right) = \left( \frac{p}{5} \right). \]

It results that

\[ F_{2p} = \left( \frac{5}{p} \right) \pmod{p}. \]

Using the properties of Legendre’ symbol it results:

\[ p \left( \left\lfloor \frac{2p}{k} \right\rfloor \right) \pmod{p}. \]
3 On the box dimension of Fibonacci functions defined on \([0, 1]\)

We study the upper bound of the Fibonacci function \(f(x)\) defined on \([0, 1]\).

In order to simplify the calculation we denote by 
\(a = \frac{1 + \sqrt{5}}{2}\). If \(0 < \delta < 1\), then:

\[
m - 1 < \frac{1}{\delta} < m = \left\lfloor \frac{1}{\delta} \right\rfloor + 1 \leq \frac{1}{\delta} + 1.
\]

\[
|f((j+1)\delta) - f(j\delta)| = \left| \frac{1}{\delta} \left( a^{(j+1)\delta} - a^{j\delta} \cos(\pi(j+1)\delta) - a^{j\delta} + \frac{1}{a^{j\delta}} \cos(\pi j \delta) \right) \right|
\]
\[
= \left| \frac{1}{\delta} \left( a^{j\delta} (a^{\delta} - 1) - \frac{1}{a^{j\delta}} \left( \cos(\pi(j+1)\delta) - \cos(\pi j \delta) \right) \right) \right|.
\]

Since

\[
0 < \delta < 1, \ a > 0 \Rightarrow 1 < a^{\delta} < a \Rightarrow \left\{ \begin{array}{l}
 a^{\delta} - 1 > 0 \\
 \frac{1}{a^{\delta}} < 1
\end{array} \right.
\]

and

\[
|\cos x - \cos y| \leq |x - y|, (\forall) x, y \in \mathbb{R},
\]

then

\[
|f((j+1)\delta) - f(j\delta)| \leq \left| \frac{1}{\delta} \left( a^{j\delta} (a^{\delta} - 1) - \frac{1}{a^{j\delta}} \cos(\pi(j+1)\delta) - \cos(\pi j \delta) \right) \right|
\]
\[
\leq \left| \frac{1}{\delta} \left( a^{j\delta} (a^{\delta} - 1) - \frac{1}{a^{j\delta}} \cos(\pi(j+1)\delta) - \cos(\pi j \delta) \right) \right|
\]
\[
= \left| \frac{1}{\delta} \left( a^{j\delta} (a^{\delta} - 1) - \frac{1}{a^{j\delta}} |\cos(\pi(j+1)\delta) - \cos(\pi j \delta)| \right) \right|
\]
\[
\leq \left| \frac{1}{\delta} \left( a^{j\delta} (a^{\delta} - 1) + \frac{1}{a^{j\delta}} \pi \delta \right) \right|.
\]

Applying Lemma 1.1 we obtain:

\[
N_\delta \leq 2m + \frac{1}{\delta \sqrt{5}} \sum_{j=0}^{m-1} R_j \left[ j\delta, (j+1)\delta \right] \Leftrightarrow
\]
\[
N_\delta \leq 2m + \frac{1}{\delta \sqrt{5}} \sum_{j=0}^{m-1} a^{j\delta} (a^{\delta} - 1) + \frac{\pi \delta}{a^{j\delta}} \Leftrightarrow
\]
\[
N_\delta \leq 2m + \frac{1}{\delta \sqrt{5}} \sum_{j=0}^{m-1} a^{j\delta} (a^{\delta} - 1) + \frac{\pi \delta}{a^{j\delta}} \Leftrightarrow
\]
So, the following assertion we have been proved:

**Theorem 3.1.** The upper Box dimension of the graph of the Fibonacci function defined on [0, 1] is less or equal with 1.

**References**


[5] Zhi-Wei Sun, Some sophisticated congruences involving Fibonacci numbers (http://math.nju.edu.cn/_zwsun)