Delayed Exponential Functions and their Application to Representations of Solutions of Linear Equations with Constant Coefficients and with Single Delay

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Abstract: - We develop a method for construction of solutions of linear discrete systems with constant coefficients and with single delay. Solutions are expressed with the aid of a special function called the discrete matrix delayed exponential having between every two adjoining knots the form of a polynomial. Derived formulas are applied to the construction of solutions of linear discrete systems with constant coefficients, with pure delay and with impulses. Moreover, we give a controllability method for linear discrete systems with constant coefficients and with single delay. Except for a criterion of relative controllability a control function is constructed as well. For continuous case we solve an initial-boundary problem for second-order partial differential equation and give its solution in the form of a series.

Key-Words: - Single delay, delayed exponentials, impulses, controllability, Fourier method.

1 Introduction
Theory of representations of solutions of discrete equations is an important and well established branch of the modern theory of difference equations concerned, in a broad sense, with the study of different phenomena arising in applied problems in technology, natural and social sciences. In this contribution we discuss new direction in investigation of linear systems. Delayed matrix exponential is used for representation of solutions of linear systems of discrete equations with constant coefficients and with single delay. In the paper we give formulas for representation of solutions of linear systems and their modification for systems with impulses. Moreover, we give a controllability method for linear discrete systems with constant coefficients and with single delay. Except for a criterion of relative controllability a control function is constructed as well. For continuous case we solve an initial-boundary problem for second-order partial differential equation and give its solution in the form of a series.

Some of results of Parts 2–5 were previously published and we refer to [1]–[4]. Result of the Part 6 is new.

2 Delayed Matrix Exponential
Throughout this paper we use following notation: for integers $s, q, s \leq q$ we define $\mathbb{Z}^q_s := \{s, s+1, \ldots, q\}$, where possibility $s = -\infty$ or $q = \infty$ is admitted, too. Using notation $\mathbb{Z}^q_s$ or another one with couple of integers $s, q$, we suppose $s \leq q$.

Let us consider a linear non-homogeneous discrete system with single delay

$$\Delta x(k) = Bx(k-m) + f(k),$$

where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}^\infty_0$, $B$ is a constant $n \times n$ matrix, $\Delta x(k) = x(k+1) - x(k)$, $x: \mathbb{Z}^\infty_{-m} \to \mathbb{R}^n$ is unknown solution and $f: \mathbb{Z}^\infty_0 \to \mathbb{R}$ is the input scalar function. Together with (1) we consider an initial (Cauchy) problem

$$x(k) = \varphi(k)$$

where $\varphi(k)$ is an initial condition.
with a given initial function \( \varphi: \mathbb{Z}_m^0 \to \mathbb{R}^n \).

The existence and uniqueness of solution of initial problem (1), (2) on \( \mathbb{Z}_m^\infty \) is obvious. We recall that solution \( x: \mathbb{Z}_m^\infty \to \mathbb{R}^n \) of initial Cauchy problem (1), (2) is defined as an infinite sequence \( \{ \varphi(-m), \varphi(-m + 1), \ldots, \varphi(0), x(1), x(2), \ldots, x(k), \ldots \} \) such that for any \( k \in \mathbb{Z}_m^\infty \) equality (1) holds. Similarly we explain above properties for more general linear discrete systems. Throughout the paper we adopt customary notations \( \sum_{i=k+s}^m \circ (i) = 0 \) and \( \prod_{i=k+s}^m \circ (i) = 1 \), where \( k \) is an integer, \( s \) is a positive integer and \( \circ \) denotes the function considered irrespective on the fact if it is for indicated arguments defined or not.

The delayed exponential is a useful tool for formalizing of computation of initial problems for systems of linear equations with constant coefficients since usually used method of steps (being nevertheless hidden in the notion of delayed exponential) gives unwieldy formulas. Discrete systems containing only one delay are often called systems with single delay. We define a discrete matrix function \( \exp_m(Bk) \) called the discrete matrix delayed exponential of an \( n \times n \) constant matrix \( B \):

\[
\exp_m(Bk) = \begin{cases} 
\Theta & \text{if } k \in \mathbb{Z}_m^{-m-1}, \\
I & \text{if } k \in \mathbb{Z}_m^0, \\
\ldots \\
I + B \cdot \binom{k - m}{1} + B^2 \cdot \binom{k - m}{2} + \cdots + B^\ell \cdot \binom{k - (\ell - 1)m}{\ell} & \text{if } k \in \mathbb{Z}_m^{\ell(m+1)} \\
\ell = 0, 1, 2, \ldots 
\end{cases}
\]

where \( \Theta \) is \( n \times n \) null matrix.

**Theorem 1** Let \( B \) be a constant \( n \times n \) matrix. Then for \( k \in \mathbb{Z}_m^{-m} \)

\[
\Delta \exp_m(Bk) = B \exp_m(B(k-m)).
\]

**3 Linear Discrete Systems**

In this part we use the discrete delayed matrix exponential to derive representations of solutions of linear discrete systems with constant coefficients.

**3.1 Homogeneous Systems**

Set \( f \equiv 0 \) in (1) and consider homogeneous system

\[
\Delta x(k) = Bx(k - m). \tag{3}
\]

**Theorem 2** Let \( B \) be a constant \( n \times n \) matrix. Then solution of the problem (3), (2) can be expressed as

\[
x(k) = e_m^{Bk} \varphi(-m) + \sum_{j=-m+1}^0 e_m^{B(k-m-j)} \Delta \varphi(j-1), \tag{4}
\]

where \( k \in \mathbb{Z}_m^\infty \).

**3.2 Non-homogeneous Systems**

Now we consider the initial problem (1), (2) where \( \varphi(k) = 0, k \in \mathbb{Z}_m^0 \). Using discrete delayed matrix exponential it is possible to determine its particular solution \( x_p(k), k \in \mathbb{Z}_m^\infty \).

**Theorem 3** Solution \( x = x_p(k) \) of the initial Cauchy problem (1), (2) where \( \varphi(k) = 0, k \in \mathbb{Z}_m^0 \) can be represented on \( \mathbb{Z}_m^\infty \) in the form

\[
x_p(k) = \sum_{j=1}^k e_m^{B(k-m-j)} f(j-1).
\]

Collecting results of Theorem 2 and Theorem 3 we get immediately

**Theorem 4** Solution \( x = x(k) \) of the initial Cauchy problem (1), (2) can be on \( \mathbb{Z}_m^\infty \) represented in the form

\[
x(k) = e_m^{Bk} \varphi(-m) + \sum_{j=-m+1}^0 e_m^{B(k-m-j)} \Delta \varphi(j-1) + \sum_{j=1}^k e_m^{B(k-m-j)} f(j-1). \tag{5}
\]

**4 Linear Discrete Systems with Impulses**

In this part we use the discrete delayed matrix exponential to derive representations of solutions of linear discrete systems with single delay and with impulses. Cases of only one impulse focused in a given interval and impulses acting at all points of a given interval are considered.

Consider initial Cauchy problem

\[
\Delta x(k) = Bx(k - m), \quad k \in \mathbb{Z}_m^\infty, \tag{5}
\]

\[
x(k) = \varphi(k), \quad k \in \mathbb{Z}_m^0 \tag{6}
\]

where \( m \geq 1 \) is a fixed integer. We add impulses \( J_i \in \mathbb{R}^n \) to \( x \) at points having a form \((i-1)(m+1)+p\) where the index \( i \geq 1 \) is defined as \( i = \lfloor \frac{k+m}{m+1} \rfloor \) for every \( k \in \mathbb{Z}_m^\infty \), i.e., we set

\[
x((i-1)(m+1)+p) = x((i-1)(m+1)+p-0) + J_i \tag{7}
\]
where $p = 1, 2, 3, \ldots, m + 1$ and investigate the solution of problem (5) – (7). Further we add impulses $J_k \in \mathbb{R}^n$ to $x$ at each point $k \in \mathbb{Z}_0^\infty$, i.e., we set
\begin{equation}
x(k + 1) = x(k + 1 - 0) + J_{k+1}
\end{equation}
and investigate the solution of the problem (5), (6), (8).

**Theorem 5** Let $B$ be a constant $n \times n$ matrix, $m$ be a fixed integer, $J_i \in \mathbb{R}^n$, $i \geq 1$, $i = \lfloor \frac{k+1}{m+1} \rfloor$. Then the solution of the initial Cauchy problem with impulses (5) – (7) can be expressed in the form:
\begin{equation}
x(k) = e^{Bk} \varphi(-m) \\
+ \sum_{j=-m+1}^{0} e^{B(k-m-j)} \Delta \varphi(j - 1) \\
+ \sum_{q=1}^{k} J_q e^{B(k-(p-1)-q(m+1))}
\end{equation}
where $k \in \mathbb{Z}_0^\infty$.

**Theorem 6** Let $B$ be a constant $n \times n$ matrix, $m$ be a fixed integer, $J_i \in \mathbb{R}^n$. Then the solution of the initial Cauchy problem with impulses (5), (6), (8) can be expressed in the form:
\begin{equation}
x(k) = e^{Bk} \varphi(-m) \\
+ \sum_{j=-m+1}^{0} e^{B(k-m-j)} \Delta \varphi(j - 1) \\
+ \sum_{i=1}^{k} J_i e^{B(k-(i+m))}
\end{equation}
where $k \in \mathbb{Z}_0^\infty$.

### 5 Controllability of Linear Discrete Equations

One of rapidly developed directions is investigation of controllability of linear discrete systems with delay. Such research is important due to numerous applications of mathematical models in natural sciences, economics and engineering. Let us consider a discrete system with pure delay
\begin{equation}
\Delta x(k) = Bx(k - m) + bu(k),
\end{equation}
where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, $b \in \mathbb{R}^n$ is a given nonzero vector and $u: \mathbb{Z}_0^\infty \rightarrow \mathbb{R}$ is the input scalar function. Together with (9) we consider an initial (Cauchy) problem (2).

System (9) is called relatively controllable, if for any initial function $\varphi: \mathbb{Z}_0^{-m} \rightarrow \mathbb{R}^n$, any finite terminal state $x = x^* \in \mathbb{R}^n$, and any finite terminal point $k_1$ greater or equal than a fixed integer $k^* \in \mathbb{Z}_1^\infty$ there exists a discrete function $u^*: \mathbb{Z}_0^{k_1-1} \rightarrow \mathbb{R}$ such that the system (9) with the input $u = u^*$ has a solution $x^*: \mathbb{Z}_0^{-m} \rightarrow \mathbb{R}^n$ such that $x^*(k_1) = x^*$ and $x^*(k) = \varphi(k)$ if $k \in \mathbb{Z}_0^{-m}$.

Let us define an auxiliary $n \times n$ matrix
\begin{equation}
S \overset{\text{def}}{=} (b, Bb, B^2b, \ldots, B^{n-1}b)
\end{equation}
and the vector
\begin{equation}
\xi \overset{\text{def}}{=} x^* - e^{Bk} \varphi(-m) \\
- \sum_{j=-m+1}^{0} e^{B(k-m-j)} \Delta \varphi(j - 1).
\end{equation}

**Theorem 7** Problem
\begin{equation}
\Delta x(k) = Bx(k - m) + bu(k), \ k \in \mathbb{Z}_0^{k_1-1},
\end{equation}
\begin{equation}
x(k) = \varphi(k), \ k \in \mathbb{Z}_0^{-m},
\end{equation}
\begin{equation}
x(k_1) = x^*.
\end{equation}
is relatively controllable if and only if
\begin{equation}
\text{rank} \ S = n, \ k_1 \geq (n-1)(m+1) + 1
\end{equation}
hold simultaneously.

A relevant control function is given in the following result.

**Theorem 8** Let the conditions of relative controllability (13) be valid. Then a control function $u = u^*$ for the problem (10)-(12) can be expressed in the form
\begin{equation}
u^*(k) = b^T \left(e^{B(k_1-k-1)} \right)^T G^{-1} \xi
\end{equation}
where the matrix
\begin{equation}
G \overset{\text{def}}{=} \sum_{j=1}^{k_1} e^{B(k_1-j)b} b^T \left(e^{B(k_1-j)} \right)^T
\end{equation}
is nonsingular.

### 6 Application of Delayed Exponential Function to Partial Differential Equations

Now we will consider autonomous linear partial homogeneous differential equations of the second order of parabolic type
\begin{equation}
\frac{\partial \xi(x, t)}{\partial t} = \lambda \frac{\partial^2 \xi(x, t)}{\partial x^2} + \sigma \xi(x, t),
\end{equation}
where $\lambda, \sigma \in \mathbb{R}$, $0 \leq x \leq l$, $\ell > 0$ and $t \geq -\tau, \tau > 0$. We assume that $\xi$ satisfies boundary conditions
\begin{equation}
\xi(0, t) = \mu_1(t), \ \xi(l, t) = \mu_2(t),
\end{equation}
where $\mu_1, \mu_2 \in \mathbb{R}$. Then there exists a discrete function $\xi^*: \mathbb{Z}_0^{k_1-1} \rightarrow \mathbb{R}$ such that the system (14) with the input $u = u^*$ has a solution $x^*: \mathbb{Z}_0^{-m} \rightarrow \mathbb{R}^n$ such that $x^*(k_1) = x^*$ and $x^*(k) = \varphi(k)$ if $k \in \mathbb{Z}_0^{-m}$.
where \( t \geq -\tau \) and functions \( \mu_i : [\tau, \infty) \to \mathbb{R}, \ i = 1, 2 \) are continuously differentiable, and initial condition

\[
\xi (x, t) = \varphi (x, t) \tag{16}
\]

where \( 0 \leq x \leq l \), \( -\tau \leq t \leq 0 \) and function \( \varphi : [0, l] \times (-\tau, 0] \to \mathbb{R} \) is continuously differentiable. We assume as well that initial and boundary conditions satisfy compatibility conditions \( \mu_1 (t) = \varphi (0, t), \ \mu_2 (t) = \varphi (l, t) \) where \( -\tau \leq t \leq 0 \).

6.1 Delayed Exponential Function and its Properties

Solution of the problem (14)–(16) will be performed by classical method of separation of variables (Fourier method). Nevertheless, due to delayed argument arise complications how to solve analytically auxiliary initial Cauchy problems for first-order linear differential equations with a single delay. We overcome this circumstance using a special function called a delayed exponential. Here we give definition the delayed exponential, a necessary overview of its properties and solution of initial problem for first-order homogeneous and non-homogeneous linear differential equations with a single delay.

**Definition 1** Let \( b \in \mathbb{R} \). The delayed exponential function \( \text{exp}_\tau \{ b, t \} : \mathbb{R} \to \mathbb{R} \) is a continuous on \( \mathbb{R} \setminus \{ -\tau \} \) function defined as

\[
\exp_\tau \{ b, t \} = \begin{cases} 
0, & -\infty < t < -\tau, \\
1, & -\tau \leq t < 0, \\
1 + \frac{b t}{1!} + \frac{b^2 (t - \tau)^2}{2!} + \cdots + \frac{b^k (t - (k - 1) \tau)^k}{k!}, & (k - 1) \tau \leq t < k \tau \\
\cdots
\end{cases}
\]

where \( k = 0, 1, 2, \ldots \).

We will consider the linear non-homogeneous differential equation with a single delay

\[
\dot{x} (t) = ax (t) + bx (t - \tau) \tag{17}
\]

where \( a, b \in \mathbb{R} \), together with initial Cauchy condition

\[
x (t) = \beta (t), \ t \in [-\tau, 0]. \tag{18}
\]

**Theorem 9** Let the function \( \beta \) in (18) be continuously differentiable. Then the unique solution of the initial Cauchy problem (17), (18) can be represented as

\[
x (t) = \exp_\tau \{ b_1, t \} e^{a(t+\tau)} \beta (-\tau) + \int_{-\tau}^{0} \exp_\tau \{ b_1, t - t - s \} e^{a(t-s)} \left[ \beta' (s) - a \beta (s) \right] ds 
\]

where \( b_1 = be^{-a\tau} \) and \( t \in [-\tau, \infty) \).

Let a linear non-homogeneous delay equation with a single delay

\[
\dot{x} (t) = ax (t) + bx (t - \tau) + f (t) \tag{19}
\]

be given, where \( a, b \in \mathbb{R} \) and \( f : [0, \infty) \to \mathbb{R} \). We consider the Cauchy problem with a zero initial condition

\[
x (t) = 0, \ t \in [-\tau, 0]. \tag{20}
\]

**Theorem 10** The unique solution of the problem (19), (20) is given by the formula

\[
x (t) = \int_{0}^{t} \exp_\tau \{ b_1, t - t - s \} e^{a(t-s)} f (s) ds \tag{21}
\]

where \( b_1 = be^{-a\tau} \).

6.2 Construction of Solution of Equation (14)

We will consider the equation (14) with the boundary conditions (15) and the initial condition (16). We will construct solution in the form

\[
\xi (x, t) = \xi_0 (x, t) + \xi_1 (x, t) \tag{22}
\]

where \( \xi_0 (x, t) \) is a solution of (homogeneous) equation (14) with zero boundary conditions

\[
\xi_0 (0, t) = 0, \ \xi_0 (l, t) = 0, \ t \geq -\tau
\]

and with nonzero initial condition

\[
\xi_0 (x, t) = \Phi (x, t) := \varphi (x, t) - \mu_1 (t)
\]

\[
- \frac{x}{l} \left[ \mu_2 (t) - \mu_1 (t) \right], \ (x, t) \in [0, l] \times [-\tau, 0], \tag{23}
\]

and \( \xi_1 (x, t) \) is a solution of a non-homogeneous equation

\[
\frac{\partial \xi (x, t)}{\partial t} = \lambda \frac{\partial^2 \xi (x, t - \tau)}{\partial x^2} + \sigma \xi (x, t) + F (x, t) \tag{24}
\]

where

\[
F (x, t) := \sigma \left[ \mu_1 (t) + \frac{x}{l} \mu_2 (t) - \mu_1 (t) \right] - \mu_1 (t) - \frac{x}{l} \left[ \mu_2 (t) - \mu_1 (t) \right],
\]

with zero boundary conditions

\[
\xi_1 (0, t) = 0, \ \xi_1 (l, t) = 0, \ t \geq -\tau \tag{25}.
\]
and zero initial condition
\[ \xi_1(x,t) = 0, \quad 0 \leq x \leq l, \quad -\tau \leq t \leq 0. \] (26)

### 6.3 Solution of the Problem (14), (22), (23)

For finding the solution \( \xi = \xi_0(x,t) \) of the equation (14) we will use the method of separation of variables again. The solution \( \xi_0(x,t) \) is supposed to be in a form of product of two unknown functions \( X(x) \) and \( T(t) \), i.e., \( \xi_0(x,t) = X(x)T(t) \). We obtain

\[ X(x)T'(t) = \lambda X''(x)T(t - \tau) + \sigma X(x)T(t). \]

Separating variables, we have

\[ \frac{T'(t) - \sigma T(t)}{\lambda T(t - \tau)} = \frac{X''(x)}{X(x)} = -k^2, \]

where \( k \) is a constant. We consider two differential equations

\[ T'(t) - \sigma T(t) + \lambda k^2 T(t - \tau) = 0, \]
\[ X''(x) + k^2 X(x) = 0. \]

(27)

(28)

Nonzero solutions of the equation (28), which satisfy zero boundary conditions \( X(0) = 0, X(l) = 0 \) exist for the choice \( k^2 = k^2_n = (\pi n/l)^2, \ n = 1, 2, \ldots \), and are defined by formulas

\[ X(x) = X_n(x) = A_n \sin \frac{\pi n}{l} x, \quad n = 1, 2, \ldots \] (29)

where \( A_n \) are arbitrary constants. We consider the equation (27) with \( k = k_n, \ n = 1, 2, \ldots \)

\[ T'_n(t) = \sigma T_n(t) - \lambda \left( \frac{\pi n}{l} \right)^2 T_n(t - \tau). \] (30)

Each of the equations (30) represents a linear first order delay differential equation with constant coefficients. We will specify initial condition for every of equations (30). To obtain such initial conditions we will expand the corresponding initial condition \( \Phi(x,t) \) (see (23)) into Fourier series

\[ \Phi(x,t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin \frac{\pi n}{l} x, \quad x \in [0,l], \quad t \in [-\tau,0] \]

where

\[ \Phi_n(t) = \frac{2}{l} \int_0^l \varphi(s,t) \sin \frac{\pi n}{l} s \, ds \]
\[ + \frac{2}{\pi n} \left[ (-1)^n \mu_2(t) - \mu_1(t) \right]. \] (31)

We will find an analytical solution of the problem (30) with initial function (31), i.e., we will find an analytical solution of the Cauchy initial problem

\[ T'_n(t) = \sigma T_n(t) - \lambda \left( \frac{\pi n}{l} \right)^2 T_n(t - \tau), \]
\[ T_n(t) = \Phi_n(t), \quad t \in [-\tau,0], \]

for every \( n = 1, 2, \ldots \). According to formula (30) we get

\[ T_n(t) = \exp_{\tau} \{ r_1, t \} e^{\sigma(t+\tau)} \Phi_n(-\tau) \]
\[ + \int_{-\tau}^{0} \exp_{\tau} \{ r_1, t - \tau - s \} e^{\sigma(t-s)} \]
\[ \times [\Phi_n(s) - \sigma \Phi_n(s)] \, ds \]

where

\[ r_1 = -\lambda \left( \frac{\pi n}{l} \right)^2 e^{-\sigma \tau}. \]

Thus, the solution \( \xi_0(x,t) \) of the homogeneous equation (14) which satisfies zero boundary conditions (22) and nonzero initial condition (23) is in (29) we set \( A_n = 1 \)

\[ \xi_0(x,t) = \sum_{n=1}^{\infty} \left[ \exp_{\tau} \{ r_1, t \} e^{\sigma(t+\tau)} \Phi_n(-\tau) \right. \]
\[ + \int_{-\tau}^{0} \exp_{\tau} \{ r_1, t - \tau - s \} e^{\sigma(t-s)} \]
\[ \left. \times [\Phi_n(s) - \sigma \Phi_n(s)] \, ds \right] \sin \frac{\pi n}{l} x. \]

### 6.4 Non-homogeneous Equation (24)

Further we will consider the non-homogeneous equation (24) with zero boundary conditions (25) and zero initial condition (26). We will try to find the solution in a form of an expansion

\[ \xi_1(x,t) = \sum_{n=1}^{\infty} T_n(t) \frac{\pi n}{l} x \] (32)

where \( T_n^0 : [-\tau,\infty) \rightarrow \mathbb{R} \) are unknown functions. Substituting (32) in the equation (24) and equating coefficients of the same functional terms, we will obtain a system of equations:

\[ (T_n^0)'(t) = \sigma T_n^0(t) - \lambda \left( \frac{\pi n}{l} \right)^2 T_n^0(t - \tau) + f_n(t) \] (33)

where \( n = 1, 2, \ldots \) and \( f_n : [0,\infty) \rightarrow \mathbb{R} \) are Fourier coefficients of the function \( F(x,t) \), i.e.,

\[ f_n(t) = \frac{2}{l} \int_0^l F(s,t) \sin \frac{\pi n}{l} s \, ds \]
\[ = \frac{2}{\pi n} \left[ \sigma \left( (-1)^{n+1} \mu_2(t) + \mu_1(t) \right) \right. \]
\[ - \left. \left( (-1)^{n+1} \mu_2(t) + \mu_1(t) \right) \right]. \]
We assume zero initial assumptions for every equation (33). Then, by formula (21), a solution of each of the equations (33) can be written as

$$T_n(t) = \int_0^t \exp_r \{ r_1, t - \tau - s \} e^{\sigma(t-s)} f_n(s) \, ds.$$  

Hence the solution of the non-homogeneous equation (24) with zero boundary conditions and with zero initial condition is

$$\xi_1(x,t) = \sum_{n=1}^{\infty} \left[ \int_0^t \exp_r \{ r_1, t - \tau - s \} e^{\sigma(t-s)} f_n(s) \, ds \right] \times \sin \frac{\pi n}{l} x.$$  

### 6.5 Solution of the Boundary Value Problem

Now we complete particular results and we give a solution of the boundary value problem in form of a series. Using previous results, the solution of the first boundary value problem for the equation (14) is given by formula:

$$\xi(x,t) = \sum_{n=1}^{\infty} \left[ \exp_r \{ r_1, t \} e^{\sigma(t+\tau)} \Phi_n(-\tau) + \int_{-\tau}^{0} \exp_r \{ r_1, t - \tau - s \} e^{\sigma(t-s)} \times [\Phi'_n(s) - \sigma \Phi_n(s)] \, ds \right] \sin \frac{\pi n}{l} x + \mu_1(t) + \frac{x}{l} \left[ \mu_2(t) - \mu_1(t) \right].$$

The formal series is convergent if some additional conditions are valid. The proof of following theorem is omitted.

**Theorem 11** Let for a $T > 0$ and an integer $k$ such that $(k - 1) \tau \leq T < k\tau$ it holds:

$$\lim_{n \to +\infty} n^{2(k-1)} \max_{0 \leq t \leq T} |f_n(t)| = 0,$$

$$\lim_{n \to +\infty} n^{2k} \max_{-\tau \leq t \leq 0} |\Phi_n(t)| = 0.$$  

Then, within an interval $0 \leq t \leq T$, the series for $\xi$ converges absolutely and uniformly.

### 7 Conclusion

In this paper, it was shown that delayed exponential functions can successfully be applied to solving various problems of discrete and continuous equations with single delay. We discussed some of previously published our results and we derived new results in this field in Part 6 where a partial differential equation of second order was analysed. We expect that delayed functions will find new applications in various disciplines and we believe that and our approach can be modified for many of them.

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