On an Elementary Proof for a Class of Elliptic Over-determined Problems in a Doubly Connected Domain

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1 On the Saint-Venant problem in a doubly connected domain

An alternative technique for determining the configuration of domains in the class of elliptic problems when an over-specification on the boundary of the domain is prescribed is to re-formulate the problem in an equivalent integral form, where the most important ingredients in partial differential equations, maximum principles, are not used. Instead, the integral dual is then used in order to deduce that the domain in consideration is an $N$-ball. For an account on these topics we refer the reader to [2,3,6,7,8,10,11,12,16].

In their famous paper [7], L. E. Payne and P. W. Schaeffer investigated this new approach without using maximum principles. Among a variety of class of over-determined problems considered in [7] involving Green's functions as well as classical boundary value problems, they showed the following two theorems taking into consideration the following Saint-Venant problem.

Let $u$ be a classical solution of the following Saint-Venant problem

$$
\Delta u = -1 \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega, \quad (1.1)
$$

$$
\frac{\partial u}{\partial n} = -c, \quad c = \text{constant on} \; \partial \Omega, \quad (1.2)
$$

where $\Omega$ is a simply connected regular, bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $\frac{\partial u}{\partial n}$ is the exterior normal derivative of $u$ on the boundary $\partial \Omega$ which is assumed to be sufficiently regular. We cite the two statements investigated in [7] without proof.

**Theorem 1.1** Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then the following statements are equivalent

- (i) $u$ satisfies $(1.1), (1.2)$,
- (ii) $\int_{\Omega} h \, dx = c \int_{\partial \Omega} h \, ds$ for all functions $h$ harmonic in $\Omega$.

The constant $c$ in (1.3) is defined by: $c := \frac{|\partial \Omega|}{|\Omega|}$.

**Theorem 1.2** If (1.3) holds, then $\Omega$ is an $N$-ball.

The aim of this note is to extend this result for a doubly connected domain without convexity of the two boundaries for both linear and non-linear cases. In fact, we assume that $u$ is a classical solution of the Saint-Venant problem defined in a ring domain $\Omega := \Omega_0 \setminus \Omega_1$ where $\Omega_0$ and $\Omega_1$ with their boundaries are sufficiently regular, two simply connected $C^2$ domains. Here and in the following $n(x)$ denotes always the unit inner normal with respect to $\Omega$.

In [17] Willms supposed two essential and important conditions:

$$
\frac{\partial u}{\partial n} = 0 \quad \text{on} \; \partial \Omega_0, \quad (1.5)
$$

$$
\frac{\partial u}{\partial n} = -c^2 \quad \text{on} \; \partial \Omega_1, \quad (1.6)
$$

$u = b^2$ on $\partial \Omega_1$. (1.8)

In [17] Willms supposed two essential and important conditions:

$$
u < b^2, \quad \Omega_0 \text{ and } \Omega_1 \text{ are convex.} \quad (1.9)
$$
The goal of this paper is to deal with the general case without recourse to the maximum principle nor to the duality. The technique that we apply here is an elementary one. It is based on classical formula of Green and a weaker condition of the internal mean curvature.

Let us start by the harmonic boundary value problem considered in [8]. Assume that \( u \) is a classical solution of

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega_0, \\
\frac{\partial u}{\partial n} &= -c_0 \quad \text{on } \partial\Omega_1, \\
\frac{\partial u}{\partial n} &= c_1 \quad \text{on } \partial\Omega_0, \\
u &= 1 \quad \text{on } \partial\Omega_1.
\end{align*}
\]

(1.10) – (1.14)

It was shown in [8] that \( \Omega \) is an annulus of \( N \)-concentric balls, where \( \Omega_0 \) and \( \Omega_1 \) are star-shaped and the constants \( c_0 \) and \( c_1 \) verify \( 0 < c_0 < c_1 \). What we do here is to drop the convexity of the two boundaries and prove the same result by assuming only the positivity of the mean curvature of the internal boundary, denoted by \( K_1 \).

The use of auxiliary functions is an essential and crucial tool for a class of over-determined elliptic problems. What we do here is to define the combination \( \Phi \) in terms of the solution \( u \) and its gradient \( |\nabla u| \) by:

\[
\Phi := |\nabla u|^2 + \frac{2}{n} u,
\]

(1.15)

where \( u \) is classical solution of (1.10) – (1.14). With a straight forward calculation, we see easily that \( \Phi \) satisfies \( \Delta \Phi > 0 \) in \( \Omega \).

Applying the second classical formula of Green, and using the smoothness of the boundary \( \partial\Omega \), and the boundary conditions (1.11) – (1.14), we get

\[
\int_{\Omega} u \Delta \Phi \, d\mathbf{x} = \int_{\Omega} \Phi \Delta u \, d\mathbf{x} + \int_{\partial\Omega} u \frac{\partial \Phi}{\partial n} \, d\mathbf{x} - \Phi \frac{\partial u}{\partial n} \, d\mathbf{x} = c_0^2 S_0 - c_1^2 S_1 - 2 c_1^2 (N - 1) \int_{\Omega_1} K_1 \, d\mathbf{x},
\]

where \( S_0 \) and \( S_1 \) denote respectively the perimeter of \( \Omega_0 \) and \( \Omega_1 \), and \( K_1 \) denotes the mean curvature of \( \partial\Omega_1 \).

Now in view of the first classical formula of Green, we obtain

\[
c_0 S_0 = c_1 S_1.
\]

(1.17)

Combining together (1.14) and (1.15), we are conducted to

\[
\int_{\Omega} u \Delta \Phi \, d\mathbf{x} \leq c_1^2 (c_0 S_0 - c_1 S_1) - 2 c_1^2 (N - 1) \int_{\Omega_1} K_1 \, d\mathbf{x} \leq 0,
\]

(1.18)

since \( 0 < u < 1, 0 < c_0 < c_1 \) and \( K_1 > 0 \) then \( \Delta \Phi \) is non-positive in the domain \( \Omega \). Thus by (1.16) \( \Delta \Phi \) is non-positive and therefore, the combination \( \Phi \) is constant in \( \Omega \). So \( \frac{\partial \Phi}{\partial n} = 0 \) on the boundary \( \partial\Omega \) and the boundary conditions (1.9), (1.12) lead us to conclude that:

\[
K_0 = \frac{1}{N(N - 1) c_0},
\]

(1.19)

and

\[
K_1 = \frac{1}{N(N - 1) c_1}.
\]

(1.20)

We then conclude that \( \Omega_0 \) and \( \Omega_1 \) are \( N \)-concentric balls in view of Alexandrov [1].

Now we are ready to state and prove the general case in \( \mathbb{R}^N \), for \( N \geq 2 \). We investigate the non-linear over-determined elliptic problem represented by:

\[
\Delta u + f(u) = 0 \quad \text{in } \Omega,
\]

(1.21)

subject to

\[
u = 0, \quad \frac{\partial u}{\partial n} = -c_0 \quad \text{on } \partial\Omega_0,
\]

(1.22)

\[
u = 1, \quad \frac{\partial u}{\partial n} = c_1 \quad \text{on } \partial\Omega_1.
\]

(1.23)

**Theorem 1.3** Let \( u \) be a classical solution of (1.21) – (1.23), then \( \Omega \) is an annulus of \( N \)-concentric balls provided that \( f \) is a positive non-increasing function and the mean curvature \( K_1 \) of the boundary \( \partial\Omega_1 \) is positive.
Similarly to the previous result, we define \( \Phi \) by
\[
\Phi := |\nabla u|^2 + \frac{2}{N} F(u),
\]
(1.24)
where \( F(u) := \int_0^u f(s) \, ds \).

For the proof of Theorem 1.3, we distinguish two different steps. The first one consists on showing that \( \Delta \Phi \) is positive in \( \Omega \). For this purpose, we make successive partial differentiation of the function \( \Phi \) defined in (1.24). Let us compute
\[
\Phi_{,ki}(x) = 2u_{,ki}(x)u_{,i}(x) + \frac{2}{N} u_{,k}(x)f(u(x)),
\]
(1.25)
\[
\Delta \Phi(x) = 2u_{,ki}(x)u_{,ki}(x) + 2u_{,i}(x) \Delta u_{,i}(x)
+ \frac{2}{N} (u_{,kk}(x)f(u(x))
+ u_{,k}(x)u_{,k}(x)f'(u(x))),
\]
(1.26)
or equivalently, we have
\[
\Delta \Phi(x) = 2u_{,ki}(x)u_{,ki}(x)
- 2u_{,i}(x)u_{,i}(u(x))f'(u(x))
+ \frac{2}{N} u_{,i}(x)u_{,i}(x)f'(u(x)) - \frac{2}{N} f^2(u(x))
= 2u_{,ki}(x)u_{,ki}(x)
- 2u_{,i}(x)u_{,i}(x)f'(u(x)) \left( \frac{-1}{N} + 1 \right)
- \frac{2}{N} f^2(u(x))
= \frac{2}{N} [Nu_{,ki}(x)u_{,ki}(x) - (\Delta u(x))^2]
- 2u_{,i}(x)u_{,i}(x)f'(x) \left( \frac{-1}{N} + 1 \right),
\]
(1.27)
Hence by Payne’s inequality \( (Nu_{,ki}u_{,ki} - (\Delta u)^2 \geq 0 \), see Sperb’s Book) and the fact that \( f \) is a non-increasing function, we obtain
\[
c_0 S_0 - c_1 S_1 < -f(0)|\Omega| < 0,
\]
(1.29)
and the fact that \( \frac{\partial \Phi}{\partial n} = 0 \), and using the boundary conditions, we find that:
\[
K_0 = \frac{f(0)}{N(N-1)c_0},
\]
and
\[
K_1 = \frac{f(1)}{N(N-1)c_1}.
\]
To this end, we conclude that \( \Omega \) is an annulus of \( N \) concentric balls.

References:


