On Design and Control of Oscillation Using Limit Cycles

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Abstract: - In this paper, a method is presented to design and control the oscillation in circuits. In fact a mathematical approach will be presented to design the circuits called oscillator. The presented method is done in state space domain using the state variables of the circuit. The presented method is much easier than other classic methods for example to equal the open loop transfer function with minus one which is done in Laplace domain.

At the first part of the paper, the behavior of limit cycles in second-order autonomous system will be analyzed based on the behavior of some appropriate equipotential curves which will be considered around the same limit cycles. In fact two sets of equipotential curves are considered so that a set has a role as the upper band of the system trajectories and another set plays a role as the lower band. It will be shown that the stability of the limit cycles in system can be assessed using the behavior of these two set of equipotential curves.

Key-Words: - Limit cycles, oscillators, stability, autonomous system.

1 Introduction

As we know, two basic methods are essentially used to analyze the behavior of limit cycles appearing in nonlinear system [1], [2]. The first method is to draw the trajectories of the system using the softwares such as Matlab in order to detect the limit cycles of the system. It is clear that the limit cycles that is detected by this approach can be recognized as stable, unstable or semi-stable [1], [2]. The second method is based on the linearization of the system around its equilibrium point or points. The second method has two major weaknesses. Firstly, the equilibrium point of the system, which the system is linearized around it, must be on the limit cycle or in a small neighbor of it otherwise the method can not detect the real behavior of the nonlinear system. Secondly, the method can only assess the behavior of the system around the limit cycle as the form of point to point if and only if these points all locate on the limit cycle [1]. Also there are several classic methods to design oscillators. The most general and important method is based on using the equation, which the open loop transfer function of the system or electronic circuit equals minus one [3]. It is clear that the design is done in Laplace domain. As we will see, the proposed method in this paper is a method which is much easier than above method to design electronic oscillator [3], [4]. There are some researches presenting methods to analyze the stability and behavior of some special systems [5], [6], [7], [8], [9].

2 Invariant Set and Equipotential Curves

Consider the following nonlinear autonomous system
\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*} \tag{1} \]

**Definition 1:** A set is said “compact” if it is bounded and closed \[1\], \[3\].

**Definition 2:** Consider the set called \( P \) so that \( P \subset \mathbb{R}^2 \), \( P \) is said “invariant set” if the trajectories of the system beginning in \( P \) remain in it as \( t \to \infty \) \[1\], \[10\], \[11\].

**Definition 3:** Suppose that the \( X = 0 \) is the equilibrium point of the second-order autonomous system described by the equation (1) and suppose that the compact set called \( M \) includes the equilibrium point (the origin). The closed curves \( u(x_1, x_2) = C \) that belong to \( M \), enclose the equilibrium point, are called equipotential curves because for each value of \( C \) there is a closed curve with the potential of \( C \). So, all points of the \( u(x_1, x_2) = C \) have the equal potential the numerical quantity of which is \( C \) \[3\].

### 3 Limit Cycle and Stability Analysis

**Definition 4:** A limit cycle is said asymptotic stable if all trajectories in vicinity of the limit cycle converge to it as \( t \to \infty \). Otherwise the limit cycle is said semi-stable or unstable \[1\], \[11\].

**Theorem 1:** Consider the second-order autonomous system (1), suppose that no equilibrium point belongs to the compact set \( M \) which encloses the origin \( (X = 0) \). There are equipotential curves \( u_1(x_1, x_2) = C_1 \), and \( u_2(x_1, x_2) = C_2 \) with clockwise directions that belong to \( M \), enclose the origin and satisfy the following inequalities
\[
\begin{align*}
\frac{du_1(x_1, x_2)}{dt} &\geq 0 \\
\frac{du_2(x_1, x_2)}{dt} &\leq 0
\end{align*} \tag{2} \]
and the right hand of above equation can be written as
\[
\begin{align*}
\ddot{X} &= V_u \times \dot{X} \\
\dot{V}_u &\leq 0
\end{align*} \tag{12} \]

where the right hand of the above equation is the algebraic value of the vector product. So the inequality (2) can be rewritten as
\[
\ddot{X} = V_u \times \dot{X} \geq 0. \tag{11} \]

This means that the direction of the trajectories of the system (1) is to the outside of the equipotential curves \( u_1(x_1, x_2) = C_1 \) as shown in Fig. 1. In the similar manner the inequality (3) can be expressed as
\[
\ddot{X} = V_u \times \dot{X} \leq 0
\]

where \( \dot{V}_u \) is the velocity vector on the
$u_2(x_1, x_2) = C_2$ and this means that the direction of the trajectories of the system (1) is to the inside of the equipotential curves $u_1(x_1, x_2) = C_1$ as shown in Fig. 1. On the other hand there is no equilibrium points belonging to M and consequently to $\Omega(C_1, C_2)$, so there is an asymptotic stable limit cycle $L$ so that $L \subset \operatorname{int} \Omega$.

$(\Leftarrow)$ The necessary condition can similarly be proved using above geometric concepts.

Remark 1: If $\frac{du_1(x_1, x_2)}{dt} = 0$ or $\frac{du_2(x_1, x_2)}{dt} = 0$, the equipotential curve $u_1(x_1, x_2) = C_1$ or $u_2(x_1, x_2) = C_2$ itself is the limit cycle respectively.

Example 1: Consider the following nonlinear system
\[\begin{align*}
&\dot{x}_1 = x_2 - x_1 \cdot \left( x_1^4 + 2x_2^2 - 10 \right) \\
&\dot{x}_2 = -x_1^3 - 3x_2^2 \cdot (x_1^4 + 2x_2^2 - 10)
\end{align*}\]

By choosing $u_1(x_1, x_2) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2 = C_1$

, for $0 < C_1 < 2.5$, it can be seen that not only the equipotential curves $u_1(x_1, x_2) = C_1$ are closed but also $\frac{du_1(x_1, x_2)}{dt} > 0$.

Also, by choosing $u_2(x_1, x_2) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2 = C_2$

, for $2.5 < C_2$, it can be seen that not only the equipotential curves $u_2(x_1, x_2) = C_2$ are closed but also $\frac{du_2(x_1, x_2)}{dt} < 0$, so there is an asymptotic stable limit cycle locating between

\[\Omega(C_1, C_2) = \left\{ (x_1, x_2) \middle| C_1 < \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2 < C_2 \right\}\]

where $0 < C_1 < 2.5$ and $C_2 > 2.5$.

4 Control of Limit Cycle Using State Feedback

Consider the following nonlinear autonomous system
\[\begin{align*}
&\dot{x}_1 = f_1(x_1, x_2, u_1^*) \\
&\dot{x}_2 = f_2(x_1, x_2, u_2^*)
\end{align*}\]

where $u_1^*$ and $u_2^*$ are the control inputs as the form of state feedback presented by the following equations
\[\begin{align*}
&u_1^* = h_1(x_1, x_2) \\
&u_2^* = h_2(x_1, x_2)
\end{align*}\]

Now, the question is that how $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ must be chosen so that an asymptotic stable limit cycle can be added to the system (15)? Using (9), the condition $\frac{du_1(x_1, x_2)}{dt} \geq 0$ in the theorem 1 can be earned as the following inequality
\[\begin{align*}
&\frac{du_1(x_1, x_2)}{dx_1} f_1(x_1, x_2, u_1^*) + \\
&\frac{du_1(x_1, x_2)}{dx_2} f_2(x_1, x_2, u_2^*) \geq 0
\end{align*}\]

and in the similar manner the $\frac{du_2(x_1, x_2)}{dt} \leq 0$ appeared in the theorem (1), can be expressed as
\[
\frac{du_2(x_1, x_2)}{dx} = f_1(x_1, x_2, u_1^*) + \frac{du_2(x_1, x_2)}{dx} = f_2(x_1, x_2, u_2^*)
\]

(18)

The inequalities (17) and (18) give the conditions which have to be satisfied by \(u_1^*, u_2^*, u_1(x_1, x_2)\) and \(u_2(x_1, x_2)\) in order to appear an asymptotic limit cycle in the system (15).

Example 2: Consider the following system
\[
\begin{align*}
\dot{x}_1 &= x_2^7 - x_1^3 + u_1^* \\
\dot{x}_2 &= -x_1 - x_1^2 x_2 + u_2^ *
\end{align*}
\]

(19)

It is clear that the equilibrium point at the origin is asymptotic stable. Now, the state feedback lows (\(u_1^*\) and \(u_2^*\)) have to be determined so that an asymptotic stable limit cycle can be added to the resulted closed loop system. By choosing equipotential curves as
\[
\begin{align*}
u_1(x_1, x_2) &= 4x_1^2 + x_2^8 = C_1; \quad 0 < C_1 < 12 \\
u_2(x_1, x_2) &= 4x_1^2 + x_2^8 = C_2; \quad 14 < C_2
\end{align*}
\]

(20)

and replacing (20) and (21) in (17) and (18) respectively, the following inequalities are earned
\[
\begin{align*}
8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7 (-x_1 - x_1^2 x_2 + u_2^* ) &\geq 0; \\
\text{for } 0 < 4x_1^2 + x_2^8 < 12
\end{align*}
\]

(22)

and
\[
\begin{align*}
8x_1(x_2^7 - x_1^3 + u_1^*) + 8x_2^7 (-x_1 - x_1^2 x_2 + u_2^* ) &\leq 0; \\
\text{for } 14 < 4x_1^2 + x_2^8
\end{align*}
\]

(23)

It can be derived from (22) and (23) that
\[
\begin{align*}
-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* &\geq 0; \\
\text{for } 0 < 4x_1^2 + x_2^8 < 12
\end{align*}
\]

(24)

and
\[
\begin{align*}
-8x_1^4 - 8x_1^2 x_2^8 + 8x_1 u_1^* + 8x_2^7 u_2^* &\leq 0; \\
\text{for } 14 < 4x_1^2 + x_2^8
\end{align*}
\]

(25)

By choosing the state feedback lows as the following forms
\[
\begin{align*}
u_1^* &= h_1(x_1, x_2) = x_1(\beta + \frac{3}{4} x_2^8) \\
u_2^* &= h_2(x_1, x_2) = 0
\end{align*}
\]

(26)

and replacing in (24) and (25), the following inequalities are earned as the conditions to appear an asymptotic limit cycle in the system
\[
\begin{align*}
-2x_1^2 (4x_1^2 + x_2^8 - 4\beta) &\geq 0; \\
\text{for } 0 < 4x_1^2 + x_2^8 < 12
\end{align*}
\]

and
\[
\begin{align*}
-2x_1^2 (4x_1^2 + x_2^8 - 4\beta) &\leq 0; \\
\text{for } 14 < 4x_1^2 + x_2^8
\end{align*}
\]

(27)

(28)

The inequalities (27) and (28) both are satisfied, when
\[
3 \leq \beta \leq \frac{14}{3}.
\]

(29)

It also follows from the theorem 1 that the asymptotic stable limit cycle \(L\), which is added to the system (15) using state feedback, appears in the following region
\[
L = \{x_1, x_2 | 12 \leq 4x_1^2 + x_2^8 \leq 14\}.
\]

(30)

5 Design of Oscillation in Electronic Circuits

Consider the basic model of an oscillator shown in Fig. 2 including LC tank and dependent current source. The state equations of the circuit can be written as the following autonomous system
\[
\begin{align*}
\dot{x}_1 &= -\frac{1}{C} x_2 + \frac{1}{C} u_1^* \\
\dot{x}_2 &= \frac{1}{L} x_1
\end{align*}
\]

(31)

where \(x_1\) and \(x_2\) are defined as the voltage which appears across the capacitor and the current of the inductor respectively. As we see the dependent current source plays the role of the control input. By considering the equation (16) and defining the state feedback as the following equation
\[
u_1^* = f(\|x_1\|_\infty) x_1
\]

(32)

, the equation (31) can be rewritten as

Fig. 2. The basic model of an oscillator.
\[
\begin{align*}
\dot{x}_1 &= -\frac{1}{C} x_2 + \frac{1}{C} f(\|x_1\|_\infty) x_1 \\
\dot{x}_2 &= \frac{1}{L} x_1
\end{align*}
\]  

(33)

where \(\| \cdot \|_\infty\) is infinite norm and \(f(\cdot)\) is determined by the electronic elements such as BJT, MOSFET and... used to design the oscillator. Now, the equipotential curves \(u_1(x_1, x_2) = C_1\) and \(u_2(x_1, x_2) = C_2\) are considered as the circles surrounding the origin and described by the following equations

\[
\begin{align*}
u_1(x_1, x_2) &= \frac{1}{L} x_1^2 + \frac{1}{C} x_2^2 = C_1 \\
u_2(x_1, x_2) &= \frac{1}{L} x_1^2 + \frac{1}{C} x_2^2 = C_2
\end{align*}
\]

(34) (35)

where \(C_1 \leq C_2\). By checking (2) and (3) of the theorem 1, we have

\[
\frac{du_1(x_1, x_2)}{dt} = \frac{2}{LC} f(\|x_1\|_\infty) x_1^2 \geq 0
\]

(36)

\[
\frac{du_2(x_1, x_2)}{dt} = \frac{2}{LC} f(\|x_1\|_\infty) x_1^2 \leq 0.
\]

(37)

It is clear that above inequalities can not both be satisfied unless

\[
f(\|x_1\|_\infty) = 0.
\]

(38)

This is the necessary and sufficient condition to appear oscillation in the circuit.

Suppose that the oscillation appearing in the circuit is as the form of sinusoidal wave, in other word the voltage appearing across the capacitor of the tank is expressed by the following equation

\[
x_1 = V_m \cos(\omega_s t)
\]

(39)

It is clear that

\[
\omega_s = \frac{1}{\sqrt{LC}}
\]

(40)

and

\[
\|x_1\|_\infty = \sup\|V_m \cos(\omega_s t)\| = V_m
\]

(41)

so the equation (38) can be rewritten as

\[
f(\|x_1\|_\infty) = f(V_m) = 0.
\]

(42)

Example 3: Consider the Hartly and Colpitts oscillators shown in Fig. 3 and Fig. 4 and their equivalent circuit shown in Fig. 5. The equivalent circuit can be summarized and drawn again as Fig. 6. By comparing Fig. 6 and Fig. 2 and using equation (42), the necessary and sufficient condition to appear sinusoidal oscillation on the output of the Hartly and Colpitts oscillators can be earn as the following equation

\[
f(V_m) = G_m(nV_m)n - G_L - n^2(G_E + \frac{G_m(nV_m)}{\alpha}) = 0
\]

(43)

where \(G_m(\cdot)\) is the large signal transconductance of the BJT used in the oscillator. It derives from the above equation that

\[
G_m(nV_m) = \frac{G_L + n^2 G_E}{n(1 - \frac{n}{\alpha})}.
\]

Fig. 3. The Hartly oscillator.

Fig. 4. The Colpitts oscillator.

Fig. 5. The equivalent circuit of the oscillators.
From equation (43), the necessary and sufficient condition to appear oscillation in the Colpitts oscillator can be rewritten as

$$\frac{G_m(nV_m)}{g_m} = \frac{G_L + n^2G_E}{g_m n(1 - \frac{n}{\alpha})}$$

(44)

where $g_m$ is the small signal transconductance of the BJT evaluated at the quiescent point. By defining $x = \frac{nV_m}{(kT/q)}$ where $k = 1.37 \times 10^{-23} \frac{J}{K}$ is the boltzmann constant and $q = 1.6 \times 10^{-19} \text{C}$ is the charge of an electron, the equation (44) can be rewritten as

$$\frac{G_m(x)}{g_m} = \frac{G_L + n^2G_E}{g_m n(1 - \frac{n}{\alpha})}.$$  

(45)

$$\left(\frac{G_m(nV_m)n - G_L - n^2(G_E + \frac{G_m(nV_m)}{\alpha})}{x_1} \right)^{X_1} = \left(\frac{G_m(nV_m)n - G_L - n^2(G_E + \frac{G_m(nV_m)}{\alpha})}{x_2} \right)^{X_2}. \quad \text{(46)}$$

The frequency can be earned as

$$\omega_c = \frac{1}{\sqrt{LC}} = 10^7 \text{ rad/sec}$$

and

$$g_m = \frac{qI_C}{kT} = \frac{1}{56} \Omega^{-1}. \quad \text{(49)}$$

Replacing $g_m$, $n$, $G_L$, $\alpha$ and $G_E$ in equation (48) results that

$$\frac{2I_1(x)}{xI_0(x)}[1 + \ln(I_0(x))] = 0.448$$

(49)

we obtain $x \approx 3.5$ from the numerical solution of the equation (49) and replacing in $x = \frac{nV_m}{(kT/q)}$ gives $V_m = 7.9 \text{ V}$, and finally

$$V_O = 10 + 7.9 \cos(10^7 t) \text{ [v]}$$

The circuit was simulated by PROTEUS-6 software and again repeated by PSPICE-8 software, the results were the same. As we see, the result of the simulations shown in Fig. 8, validates the presented proposal.

On the other hand, we have [3]

$$\frac{G_m(x)}{g_m} = \frac{2I_1(x)}{xI_0(x)}[1 + \ln(I_0(x))] - \frac{G_L + n^2G_E}{(G_E + \frac{G_m(nV_m)}{\alpha})g_m n(1 - \frac{n}{\alpha})}. \quad \text{(48)}$$

Example 4: Consider the Colpitts oscillator shown in Fig. 7 so that $R_L = 10 \ k\Omega$, $C_1 = 1 \ nF$, $C_2 = 79 \ nF$, $R_E = 20 \ k\Omega$, $L = 10 \ \mu H$, $V_{CC} = -10 \ V$ and $\alpha = 0.99 \approx 1$.

We have

$$C = \frac{C_1C_2}{C_1 + C_2} \approx 1 \ nF, n = \frac{C_1}{C_1 + C_2} = \frac{1}{80}, \quad V_X = 9.3 \ V \text{ and } I_C = \frac{9.3 \ V}{20 \ k\Omega} = 0.465 \ mA.$$

The circuit was simulated by PROTEUS-6 software and again repeated by PSPICE-8 software, the results were the same. As we see, the result of the simulations shown in Fig. 8, validates the presented proposal.
6 Conclusion

The method presented in this paper was a mathematical method which is suitable to design and control oscillations in electronic circuits. The method was resulted from the concept of the limit cycles appearing in second order nonlinear autonomous system. In fact, it is an application of control concept in electronic circuits. Finally, as we saw the method was validated by the result of the simulations.

References: