Analysis of Nonlinear Forced Vibrations of Fractionally Damped Suspension Bridges

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Abstract: Nonlinear forced vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance, are investigated. The method of multiple time scales is used as the method of solution. The damping features are described by the fractional derivative, which is interpreted as the fractional power of the differentiation operator. The influence of the fractional parameters (orders of fractional derivatives) on the motion of the suspension bridge model is investigated.

Key–Words: Suspension bridge, nonlinear forced vibrations, fractional damping, method of multiple time scales

1 Introduction

The experimental data obtained in [1] during ambient vibration studies of the Golden Gate Bridge show that different vibrational modes feature different amplitude damping factors, and the order of smallness of these coefficients tells about low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another. Besides, as natural frequencies of vibrations increase, the corresponding damping ratios decrease.

Nonlinear free damped vibrations of suspension bridges in the cases of the one-to-one internal resonance (when the natural frequency of a certain mode of vertical vibrations is close to the natural frequency of a certain mode of torsional vibrations) and the two-to-one internal resonance (when one natural frequency is nearly twice as large as another natural frequency) have been examined in [2] when damping features of the system are prescribed by the first derivative of the displacement with respect to time. It has been shown that for the both types of the internal resonance the damping coefficient does not depend on the natural frequency of vibrations, but this result is in conflict with the experimental data presented in [1].

To lead the theoretical investigations in line with the experiment, fractional derivatives were introduced in [3] for describing the processes of internal friction proceeding in suspension combined systems at nonlinear free vibrations. The nonlinear suspension bridge model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations. The model suggested in [3] has been generalized in [4] by using two different fractional parameters for analyzing vertical and torsional modes of nonlinear damped vibrations of suspension bridges.

In the present paper, the model described in [4] is used for investigating nonlinear forced vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance. The influence of the fractional parameters (orders of the fractional derivatives) on the chaotic motion of the suspension bridge model is investigated.

2 Problem Formulation

To analyze the forced damped vibrations of suspension bridges we will use its classical scheme involving a bisymmetrical thin-walled stiffening girder connected with two suspended cables by virtue of vertical suspensions [4]. The cables are thrown over the pilons and are tensioned by anchor mechanisms. The suspensions are considered as inextensible and uniformly distributed along the stiffening girder. The cables are parabolic, and the contour of the girder’s cross-section is underformable. It is assumed that the girder’s contour translates as a rigid body vertically (in the y-axis direction) on the value of \( \eta(z,t) \) and rotates with respect to the girder’s axis (the z-axis) through the angle of \( \varphi(z,t) \). The origin of the frame of references is in
the center of gravity of the cross section.

It is known for suspension bridges [2, 5] that some natural modes belonging to different types of vibrations could be coupled with each other, i.e., the excitation of one natural mode gives rise to another one. Two modes interact more often that not, although the possibility for interaction of a greater number of modes is not ruled out.

Below we consider the case when only two modes predominate in the vibrational process, namely: the vertical \( n \)-th mode with linear natural frequency \( \omega_{0n} \), and the torsional \( m \)-th mode with the natural frequency \( \Omega_{0m} \). Under such an assumption the functions \( \eta(x, t) \) and \( \varphi(x, t) \) could be approximately defined as

\[
\eta(x, t) \sim v_n(x)x_{1n}(t), \quad \varphi(x, t) \sim \Theta_m(x)x_{2m}(t)
\]

where \( x_{1n}(t) \) and \( x_{2m}(t) \) are the generalized displacements, and \( v_n(z) \) and \( \Theta_m(z) \) are natural shapes of the two interacting modes of vibrations.

When the harmonic force \( F = \hat{F}\cos(\omega_ft) \) is applied at the center of the suspension bridge, then the equations of its forced vibrations are written in the dimensionless form as

\[
\ddot{x}_{1n} + \omega^2_{0n}x_{1n} + \beta D_{0+}^\gamma x_{1n} + a_{11}^n x_{1n}^2 + a_{12}^nm x_{2m}^2 + \frac{b_{11}n}{2}x_{1n}^2 + b_{12}nm x_{2m}^2 = \hat{F}\cos(\omega_ft)
\]

\[
\ddot{x}_{2m} + \Omega^2_{0m}x_{2m} + \beta D_{0+}^\gamma x_{2m} + a_{12}^mn x_{1n}x_{2m} + \frac{c_{11}n}{2}x_{1n}^2 + c_{12}nm x_{2m}^2 = 0
\]

where \( a_{ij}, b_{ij}, \) and \( c_{ij} \) ( \( i, j = 1, 2 \) ) are certain dimensionless coefficients which are defined in [5] (subsequently the indices \( n \) and \( m \) are omitted for ease of presentation). \( \hat{F} = \text{const} \) and \( \omega_F \) are, respectively, the amplitude and frequency of the external force, dots denote differentiation with respect to time, the terms \( \beta D_{0+}^\gamma x_{1n} \) and \( \beta D_{0+}^\gamma x_{2m} \) characterize inelastic reaction of the system, \( \beta \) is the viscosity coefficient, the fractional derivative \( D_{0+}^\gamma x \) ( \( \gamma = \gamma_1 \) or \( \gamma_2 \) ) is defined as follows [6]

\[
D_{0+}^\gamma x = \frac{d}{dt} \int_0^t \frac{x(t-t')dt'}{\Gamma(1-\gamma)t^{\gamma}} \quad (0 < \gamma \leq 1)
\]

\( \gamma \) is the order of the fractional derivative (fractional parameter), and \( \Gamma(1-\gamma) \) is the Gamma-function.

Let us consider the case of the one-to-one internal resonance, as well as suppose that the frequency of the external force is close to the natural frequency of the interacting modes, i.e.,

\[
\omega_0 \approx \Omega \approx \omega_F
\]

Since for finding the solution of Eqs. (2) we will use the method of multiple time scales, where the functions \( e^{ \pm i \omega t } \) are utilized as the main harmonic functions, then in order to carry out the calculations the following formulas will be in demand [7]

\[
D_{0+}^\gamma e^{ \pm i \omega t } = D_{+}^\gamma e^{ \pm i \omega t } + \frac{\sin \pi \gamma}{\pi} \int_0^{\infty} w e^{-ut} du \quad \text{or} \quad D_{0+}^\gamma e^{ \pm i \omega t } = (\pm i\omega)^\gamma e^{ \pm i \omega t } \quad (5)
\]

where \( D_{+}^\gamma \) is obtained from (3) changing the low limit to \(-\infty\).

It has been shown in [8] that the second term in formula (5) does not produce secular terms in the method of multiple time scales under the limitation of the zero- and first-order approximations. In other words, this term could be neglected in further consideration, and it is possible to use the approximate formula

\[
D_{0+}^\gamma e^{ \pm i \omega t } \approx D_{+}^\gamma e^{ \pm i \omega t } \quad (7)
\]

If we take into account formula (5.82) from [6]

\[
D_{0+}^\gamma e^{ \pm i \omega t } = \left( \frac{d}{dt} \right)^\gamma e^{ \pm i \omega t } \quad (8)
\]

then from the combination of (7) and (8) it follows the relationship

\[
D_{0+}^\gamma e^{ \pm i \omega t } \approx \left( \frac{d}{dt} \right)^\gamma e^{ \pm i \omega t } \quad (9)
\]

which will be used in further calculations.

### 3 Method of Solution

An approximate solution of equations (2) for small amplitudes weakly varying with time can be represented by an expansion in terms of different time scales under the assumption of weak damping \( \beta = \varepsilon \mu \) and that \( \hat{F} = \varepsilon^2 f \)

\[
x_1(t) = \varepsilon x_{11}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{12}(T_0, T_1, T_2, \ldots) + \varepsilon^3 x_{13}(T_0, T_1, T_2, \ldots) + \ldots \quad (10a)
\]

\[
x_2(t) = \varepsilon x_{21}(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_{22}(T_0, T_1, T_2, \ldots) + \varepsilon^3 x_{23}(T_0, T_1, T_2, \ldots) + \ldots \quad (10b)
\]

where \( T_n = \varepsilon^n (n = 0, 1, 2, \ldots) \) are new independent variables, \( \varepsilon \) is a small parameter which is of the same order of magnitude as the amplitudes, and \( \mu \) and \( f \) are finite values.

Considering that

\[
d/dt = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \quad (11a)
\]
Integrating Eqs. (17), we find

$$A_1(T_1, T_2) = a_1(T_2) \exp \left[ -\frac{1}{2} \mu(i\omega_0)^{\gamma-1} T_1 \right]$$

$$+ \frac{f}{2\mu(i\omega_0)^{\gamma}}$$

$$A_2(T_1, T_2) = a_2(T_2) \exp \left[ -\frac{1}{2} \mu(i\omega_0)^{\gamma-1} T_1 \right]$$

Substituting (19) in Eqs. (18) and integrating, we obtain the expressions for $x_{12}$ and $x_{22}$. Then substituting found $x_{12}$ and $x_{22}$ in Eqs. (15a) and (15b) and using the standard procedure for eliminating the secular terms, we have

$$D_2a_1 + \left[ \frac{1}{8} \mu^2(i\omega_0)^{2\gamma-3}(1 - 2\gamma_1) 
+ \frac{1}{4} \frac{f^2(a_{11}^2 - 3b_{11})}{\mu^2\omega_0^{2\gamma_1+1}} e^{-2\pi i\gamma_1} \right] a_1 = 0$$

$$D_2a_2 + \left[ \frac{1}{8} \mu^2(i\omega_0)^{2\gamma-3}(1 - 2\gamma_2) 
+ \frac{1}{4} \frac{f^2(a_{11}a_{12} - 2c_{11} - \frac{1}{4} a_{12}^2\omega_0^{-2})}{\mu^2\omega_0^{2\gamma_1+1}} e^{-2\pi i\gamma_1} \right] a_2 = 0$$

Integrating Eqs. (20) yields

$$a_1 = a_{10} \exp \left\{ T_2 \left[ -\frac{1}{8} \mu^2(1 - 2\gamma_1)(i\omega_0)^{2\gamma-3} 
- \frac{1}{4} \frac{f^2(a_{11}^2 - 3b_{11})}{\mu^2\omega_0^{2\gamma_1+1}} (i \cos 2\pi \gamma_1 + \sin 2\pi \gamma_1) \right] \right\}$$

$$+ \frac{1}{4} \frac{f^2(a_{11}^2 - 3b_{11})}{\mu^2\omega_0^{2\gamma_1+1}}$$

(21a)
where \( a_0^0 \) and \( a_0^0 \) are arbitrary constants.

Considering formulas (10), (16), (19), and (21), we finally obtain

\[
x_1 = \varepsilon \left[ 2a_0^0 e^{-\alpha_1 t} \cos \Omega_1 t \right]
\]

\[
+ \frac{f}{\mu \omega_0^2} \cos \left( \omega_0 t - \frac{\pi}{2} \gamma_1 \right) + O(\varepsilon^2) \tag{22a}
\]

\[
x_2 = \varepsilon a_2^0 e^{-\alpha_2 t} \cos \Omega_2 t + O(\varepsilon^2) \tag{22b}
\]

where

\[
\alpha_1 = \frac{1}{2} \varepsilon \mu \omega_0^{\gamma_1 - 1} \sin \left( \frac{\pi \gamma_1}{2} \right)
\]

\[
\times \left[ 1 + \frac{1}{2} \varepsilon \mu (2 \gamma_1 - 1) \omega_0^{\gamma_1 - 2} \cos \left( \frac{\pi \gamma_1}{2} \right) \right]
\]

4 Conclusion

Nonlinear forced vibrations of a suspension bridge subject to the combination of external and internal resonances have been investigated for the case when its damping features are described by the fractional derivatives. Reference to the found analytical solution (22) shows that it involves two parts: the first corresponds to the damping vibrations with damping coefficients and nonlinear frequencies dependent on the fractional parameters and describes the transient process, while the second one is nondamping in character and describes forced vibrations with the frequency of the exciting force and with the phase difference depending on the fractional parameter.

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