An Inverse Calculation for Polynomial Matrices using Regularizing Matrix

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Abstract: In this paper, we will propose a method to regularize the polynomial matrices. The regularizing matrix is closely related to the interactor matrix. As applications, we will show an inverse calculation for the polynomial matrices.

Key–Words: Linear Multivariable Systems, Polynomial Matrix Approach, Regularizing Matrix, Unimodular Matrix, Interactor Matrix

1 Introduction

The polynomial matrix approach in the analysis and synthesis of linear control systems is important scheme especially for an adaptive control. But in the course of polynomial matrix approach, it is pointed out that some operations of large dimensional real matrices are needed and the check of properness for controllers are not easy. Even several problems are solved in [1] and [2], there are some problems which are still unsolved. In this paper, we will consider a calculation method of inverse matrix for polynomial matrix using the regularizing matrix.

There is a nice relation between polynomial matrix and state space representation, which is called the structure theorem by Wolovich [3]. The theorem means that the inverse is given by using state space representation if a given polynomial matrix is row or column proper, i.e., the row (or column) leading coefficient matrix is nonsingular. If the leading coefficient matrix is not nonsingular, we should consider a method to make the leading coefficient matrix be nonsingular. For this purpose, a regularizing matrix is introduced.

The regularizing matrix is almost equivalent to an interactor matrix [4]. A derivation of the interactor is much complex. Although the algebraic equation, which the interactor must be satisfied, was shown in [5], the solution by [5] was not adequate for computer calculations. The authors proposed a solution of the equation by using Moore-Penrose pseudoinverse [6]. Since a function to calculate the pseudoinverse is available is some standard software for control engineering, the method is adequate for computer calculation.

We will consider the inverse calculation for some typical polynomial matrices. One is the unimodular of polynomial matrix and the other is a polynomial matrix having all zeros at the origin. In general, the inverse of the polynomial matrix might be a rational function matrix. But since the above matrices have no zeros or at origin, their inverse are given by polynomial matrices (in $s$ or $s^{-1}$). Therefore, the regularizing method is useful for the calculation. Some numerical examples will be also shown.

2 Regularizing Polynomial Matrix

Consider the following $q \times m$ ($q \leq m$) polynomial matrix $D(s)$:

$$D(s) = D_0 + sD_1 + \cdots + s^\mu D_\mu = DS_I^\mu m(s)$$

(1)

where

$$D = \begin{bmatrix} D_0 & D_1 & \cdots & D_\mu \end{bmatrix},$$

$$S_I^\mu m(s) = \begin{bmatrix} I_m & sI_m & \cdots & s^\mu I_m \end{bmatrix}^T.$$  

(2)

$D(s)$ is called regular if $D_\mu$ has full rank $q$. The problem considered in this section is to find a $q \times q$ nonsingular polynomial matrix $L(s)$ which makes $\mu$-th degree’s coefficient matrix of $L(s)D(s)$ be full rank and the coefficient matrices which degrees are greater than $\mu$ be zeros. $L(s)$ is called a regularizing polynomial matrix of $D(s)$. The existence of such matrix is
clear by considering the interactor for $D(s)/s^{\mu+1}$. In the
following, we will consider the direct derivation of $L(s)$ not using the interactor.
Assume that $L(s)$ has the following structure
\[ L(s) = L_0 + sL_1 + \cdots + s^w L_w \]
\[ = LS^w_s(s) \]
\[ L = \begin{bmatrix} L_0 & L_1 & \cdots & L_w \end{bmatrix} \]
where the integer $w$ will be defined later. Then, $L(s)D(s)$ can be written by
\[ L(s)D(s) = LS^w_s(s)(D_0 + s D_1 + \cdots + s^\mu D_\mu)S^w_{I_m}(s) \]
\[ = L \begin{bmatrix} D_0 & D_1 & \cdots & D_w & \cdots & D_\mu & 0 & \cdots & 0 \\ 0 & D_0 & \cdots & D_{w-1} & \cdots & D_{\mu-1} & D_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} S^w_{I_m}(s), \]
where $D_{\mu-w} = 0$ if $\mu - w < 0$. Assume that the $\mu$-th degree’s coefficient matrix of $L(s)D(s)$ is $K \in \mathbb{R}^{k \times m}$. If $L(s)$ is the regularizing matrix, then the following equality must hold from the above relation:
\[ LT_w = J \]
where
\[ T_w = \begin{bmatrix} D_\mu & 0 & \cdots & 0 \\ D_{\mu-1} & D_\mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{\mu-w} & D_{\mu-w+1} & \cdots & D_\mu \end{bmatrix}, \]
\[ J = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}. \]
Considering the structure of $J$, set
\[ L = JT_w^\dagger = K T_w^\dagger (1 : m,:), \]
where $T_w^\dagger (1 : m,:)$ denote the submatrix constituted of the first $m$-th rows of $T_w^\dagger$. Substituting the above equation to eqn.(5),
\[ KT_w^\dagger (1 : m,:) T_w = J. \]
Define $A$ by
\[ A = T_w^\dagger (1 : m,:) \begin{bmatrix} D_\mu \\ D_{\mu-1} \\ \vdots \\ D_{\mu-w} \end{bmatrix}, \]
the first $m$-th columns of eqn.(8) can be written by
\[ KA = K. \]
That is, if eqn.(5) is solvable, its special solution is given by eqn.(7) and $K$ must satisfy eqn.(10). Let $U \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} V^T$ denote the singular value decomposition (SVD) of $T_w$ using some nonsingular matrix $\Gamma$ and unitary matrices $U$ and $V$. Then, $T_w^\dagger$ is given by
\[ T_w^\dagger = V \begin{bmatrix} \Gamma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T \]
and
\[ T_w^\dagger T_w = V \begin{bmatrix} I \Gamma & 0 \\ 0 & 0 \end{bmatrix} V^T. \]
Therefore, $A$ can be written by
\[ A = V(1 : m,:) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^T (1 : m,:) \geq 0. \]
Eqn.(10) means that $K$ is the left eigenvectors of $A$ which correspond to the eigenvalues at $\lambda = 1$. Since $A$ is a real symmetric matrix, the geometric multiplicity of the eigenvalue one in $A$ equals to the algebraic multiplicity. Thus we can find a set of linearly independent eigenvectors for the eigenvalue one. Therefore,
1. $w$ is the least integer when $\lambda$ has $p$ multiple eigenvalue at $\lambda = 1$.
2. $K$ is constituted of corresponding left eigenvectors.

Example 1 Consider the following polynomial matrix:
\[ D(s) = \begin{bmatrix} s + 1 & s + 2 & s + 3 \\ s + 4 & s + 5 & s + 6 \end{bmatrix}. \]
For the above case, $q = 2$, $m = 3$ and $\mu = 1$. $D_0$ and $D_1$ are given by
\[ D_0 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \]
Setting $w = 2$, $T_2$ is given by
\[ T_2 = \begin{bmatrix} D_1 & 0 & 0 \\ D_0 & D_1 & 0 \\ 0 & D_0 & D_1 \end{bmatrix}. \]
and then $A$ is given by
\[ A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \]
which has the eigenvalue at \( \lambda = 1 \) with multiplicity 2 = \( p \). The left eigenvectors of \( A \) corresponding to \( \lambda = 1 \) are given by \([1 \ 0 \ -1]\) and \([0 \ 1 \ 2]\) and thus \( K \) is given by

\[
K = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.
\]

Therefore, \( L(s) \) can be calculated by

\[
L(s) = \begin{bmatrix} K & 0 & 0 \end{bmatrix} T_2 S^2(s) = \begin{bmatrix} .5385 & .5385 \\ -.3846 & -.3846 \end{bmatrix} + s \begin{bmatrix} -1.3077 & .3077 \\ 1.0769 & -.0769 \end{bmatrix} + \frac{s^2}{3} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

## 3 Inverse of Polynomial Matrix

### 3.1 General Case

Let \( D(s) \) and \( L(s) \) denote a polynomial matrix and its regularizing polynomial matrix. Then, the inverse of \( D(s) \) is given by

\[
D^{-1}(s) = \{L(s)D(s)\}^{-1}L(s). \tag{12}
\]

From the structure theorem [3], the observability canonical realization of \( \{L(s)D(s)\}^{-1}L(s) \) can be easily obtained and it gives the inverse of \( D^{-1}(s) \).

### 3.2 Unimodular Matrix

Let \( U(z) \) and \( V(s) \) denote a unimodular matrix of polynomial matrix and its inverse matrix. Since \( V(s)U(s) = I \), the following equation must hold:

\[
VT(U)_w = J, \tag{13}
\]

where

\[
V := [V_0 \ V_1 \ \cdots \ V_w], \quad J := [I \ 0 \ \cdots \ 0],
\]

\[
T(U)_w := \begin{bmatrix} U_0 & 0 & \cdots & 0 \\ U_1 & U_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & U_{w-1} U_w \end{bmatrix},
\]

\[
U(s) = U_0 + sU_1 + \cdots + s^w U_w,
\]

\[
V(s) = V_1 + sV_1 + \cdots + s^w V_w.
\]

Then, the coefficient matrix \( V \) of \( V(s) \) is given by

\[
V = J T(U)_w. \tag{14}
\]

### Example 2

Consider the following unimodular matrix \( U(s) \).

\[
U(s) = \begin{bmatrix} s + 1 & s + 2 \\ s + 3 & s + 4 \end{bmatrix} = U_0 + sU_1, \quad U_0 = \begin{bmatrix} 12 \\ 34 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 11 \\ 11 \end{bmatrix}.
\]

In this case, we can confirm \( w = 2 \) and thus \( T(U)_2 \) is given by

\[
T(U)_2 = \begin{bmatrix} U_1 & 0 & 0 \\ U_0 U_1 & 0 & 0 U_0 U_1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Thus, \( V \) is given by

\[
V = JT(U)_2 \begin{bmatrix} 0.75 & 0.75 & -1.25 & 0.25 & -0.5 & 0.5 \\ -0.5 & -0.5 & 1 & 0 & 0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.75 & -1.25 & 0.25 & -0.5 & 0.5 \\ -0.5 & -0.5 & 1 & 0 & 0.5 & -0.5 \end{bmatrix},
\]

that is,

\[
V(s) = \begin{bmatrix} -0.5s^2 - 1.25s + 0.75 \ 0.5s^2 + s - 0.5 \end{bmatrix}.
\]

### 3.3 Having All Zeros at Origin

Let \( D(s) \) denote the polynomial matrix having all zeros at the origin. Then, its inverse matrix \( D^{-1}(s) \) can be written by

\[
D(s) = D_0 + s^{-1}D_1 + \cdots + s^{-w}D_w. \tag{15}
\]

Since \( D(s)D(s) = I \), the following equation must hold:

\[
\bar{D}T(D)_w = J, \tag{16}
\]

where

\[
D := [D_w \ D_{w-1} \ \cdots \ D_0], \quad J := [0 \ \cdots \ I \ 0 \ \cdots \ 0],
\]
Thus, given by

\[ w = \frac{1}{s + 1}. \]

In this case, we can confirm \( w = 2 \) and thus \( T(D)_2 \) is given by

\[
T(D)_2 := \begin{bmatrix}
D_0 \cdots D_\mu & 0 & \cdots & 0 \\
0 & D_0 \cdots D_\mu & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & D_0 \cdots D_\mu
\end{bmatrix},
\]

\[ D(s) = D_0 + sD_1 + \cdots + s^\mu D_\mu. \]

Then, the coefficient matrix \( \bar{D} \) of \( \bar{D}(s) \) is given by

\[ \bar{D} = JT^\dagger(D)_w. \quad (17) \]

Example 3 Consider the following transfer function matrix \( G(s) \):

\[
G(s) = \begin{bmatrix}
1 & 1 \\
\frac{1}{s + 1} & \frac{1}{s + 2} \\
\frac{1}{s + 3} & \frac{1}{s + 4}
\end{bmatrix}.
\]

For the above transfer function matrix, an interactor \( L(s) \) proposed in [6] is given by

\[
L(s) = \frac{s}{2} \begin{bmatrix}
s^2 + 0.5s + 1.5 & -s^2 - 2.5s + 1.5 \\
-s^2 - 1 & s^2 + 2s - 1
\end{bmatrix},
\]

which has all zeros at the origin. Set

\[
D_0 = \begin{bmatrix}
0.75 & 0.75 \\
-0.5 & -0.5
\end{bmatrix},
\]

\[
D_1 = \begin{bmatrix}
0.25 & -1.25 \\
0 & 1
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
0.5 & -0.5 \\
-0.5 & 0.5
\end{bmatrix}.
\]

In this case, we can confirm \( w = 2 \) and thus \( T(D)_2 \) is given by

\[
T(D)_2 := \begin{bmatrix}
D_0 & D_1 & D_2 & 0 & 0 \\
0 & D_0 & D_1 & D_2 & 0 \\
0 & 0 & D_0 & D_1 & D_2
\end{bmatrix}.
\]

Thus, \( \bar{D} \) is given by

\[
\bar{D} = JT^\dagger(D)_w
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0.5 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
-1 & 1 & 1 & 2 & 2.5 & 1 \\
1 & 1.5 & 0 & 0.5 & 1 & 1 \\
0 & 0 & -1 & 1 & 2 & 2.5 \\
0 & 0 & 1 & 1.5 & 0 & 0.5 \\
0 & 0 & 0 & 0 & -1 & -1.5 \\
0 & 0 & 0 & 0 & 1 & 1.5
\end{bmatrix}
\]

\[ D^{-1}(s) = \begin{bmatrix}
s^2 + 2s - 1 & s^2 + 2s - 1.5 \\
s^2 + 1 & s^2 + 0.5s + 1.5
\end{bmatrix}. \]

4 Conclusions

In this paper, a regularizing polynomial matrix was proposed. A calculation method of the regularizing matrix was proposed. This method is easy and adequate for computer calculations. As application, the inverse calculation of polynomial matrix was shown. Numerical examples were also presented to confirm the effectiveness of the proposed method.

References:


