On the Equivalence of Control Systems on the Orthogonal Group $SO(4)$

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Abstract: We investigate a certain class of left-invariant control systems evolving on the Lie group $SO(4)$. Two such systems are $\mathcal{L}$-equivalent provided their traces are related by a Lie algebra automorphism. We produce structural results regarding $\mathcal{L}$-equivalence of all homogeneous control affine systems on $SO(4)$. An illustrative example is provided.

Key–Words: Invariant control affine system, $\mathcal{L}$-equivalence, feedback equivalence, Lie algebra automorphism.

1 Introduction

Invariant control systems are (smooth, nonlinear) control systems evolving on (real, finite dimensional) Lie groups with dynamics invariant under translations. Such control systems have been studied by a number of authors over the last few decades (see, e.g., [2], [8], [9]). In order to understand the local geometry of (nonlinear) control systems it is useful to introduce natural equivalence relations. The most important (and useful) equivalence relations are state space reducible natural equivalence relations. The most important (and useful) equivalence relations are state space reducible natural equivalence relations. Also, consider the $\mathcal{L}$-equivalence (of control systems) on Lie groups that has been investigated by Biggs and Remsing [4]. (For some concrete cases, see [1], [5].)

In this paper we consider only left-invariant control affine systems, evolving on the (six-dimensional) orthogonal group $SO(4)$. These systems have the form

$$\dot{g} = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \quad 1 \leq \ell \leq 6$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}(4)$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) Some interesting results concerning (invariant) optimal control problems on $SO(4)$ have been obtained in recent years (see, e.g., [3], [6]). We study the structure of such homogeneous systems, with respect to detached feedback equivalence. Specifically, we prove that any homogeneous system is $\mathcal{L}$-equivalent to one of a list of equivalence representatives.

A few remarks pertaining to the classification of systems, under $\mathcal{L}$-equivalence, conclude the paper.

2 Invariant control systems

A left-invariant control affine system $\Sigma$ is a control system of the form

$$\dot{g} = g(\Xi(1, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell)$$

where $g \in G$ and $u \in \mathbb{R}^\ell$. Here $G$ is a (real, finite dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$. Also, the parametrization map $\Xi(1, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent). Note that the dynamics $\Xi : G \times \mathbb{R}^\ell \to TG$ is invariant under left translations i.e., $\Xi(g, u) = g \Xi(1, u)$. Such a system is denoted by $\Sigma = (G, \Xi)$.

The trace of $\Sigma$, $\Gamma = \mathrm{im} \Xi(1, \cdot) = A + \langle B_1, \ldots, B_\ell \rangle$, is an affine subspace of $\mathfrak{g}$. A system is called homogeneous if $A \in \langle B_1, \ldots, B_\ell \rangle$, and inhomogeneous otherwise. $\Sigma$ has full rank if the Lie algebra generated by its trace coincides with $\mathfrak{g}$.

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A trajectory for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to G$ such that $\dot{g}(t) = g(t) \Xi(1, u(t))$ for almost every $t \in [0, T]$.

Let $\Sigma = (G, \Xi)$ and $\Sigma' = (G, \Xi')$ be two systems on $G$. We say that $\Sigma$ and $\Sigma'$ are (locally) detached feedback equivalent if there exist open neighborhoods $N$ and $N'$ of (the unit element) $1$ and a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(1) = 1$ and $T_g \Phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. Two detached feedback equivalent systems have the same trajectories (up to a diffeomorphism in the state space).
which are parametrized differently by admissible controls. We say that the systems \( \Sigma \) and \( \Sigma' \) (with traces \( \Gamma \subseteq g \) and \( \Gamma' \subseteq g, \) respectively) are \( \mathcal{L} \)-equivalent if there exists a Lie algebra automorphism \( \psi : g \to g \) such that \( \psi \cdot \Gamma = \Gamma' \). Suppose \( \Sigma \) and \( \Sigma' \) have full rank.

**Proposition 1 ([4])** \( \Sigma \) and \( \Sigma' \) are detached feedback equivalent if and only if they are \( \mathcal{L} \)-equivalent.

**Remark 2** Suppose \( \Sigma \) and \( \Sigma' \) do not have full rank. If the systems are \( \mathcal{L} \)-equivalent, then they are detached feedback equivalent, but not conversely.

### 3 The orthogonal group \( \text{SO}(4) \)

The orthogonal group \( \text{SO}(4) = \{ g \in \text{GL}(4, \mathbb{R}) : g^{-1}g = 1, \det g = 1 \} \) is a six-dimensional, non-commutative, semisimple, compact Lie group. Its Lie algebra \( \mathfrak{so}(4) = \{ A \in \mathbb{R}^{4 \times 4} : A^T + A = 0 \} \) is isomorphic to \( \mathfrak{so}(3) \oplus \mathfrak{so}(3) \). Let

\[
E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

be the standard (ordered) basis for \( \mathfrak{so}(3) \). The map \( \zeta : \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4) \), given by

\[
\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}
\]

is a Lie algebra isomorphism. The natural basis of \( \mathfrak{so}(4) \) is given by

\[
E_i = \zeta \cdot (E_i, 0) \quad i = 1, 2, 3
\]

\[
E_j = \zeta \cdot (0, E_{j-3}) \quad j = 4, 5, 6.
\]

The commutator table for \( \mathfrak{so}(4) \) is given below.

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<tr>
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<th>( E_1 )</th>
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**Lemma 3** The group of inner automorphisms of \( \mathfrak{so}(4) \) is given by

\[
\text{Int}(\mathfrak{so}(4)) = \left\{ \psi_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : \psi_1, \psi_2 \in \text{SO}(3) \right\}.
\]

For convenience we will identify an inner automorphism \( \psi = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \) with the pair \( (\psi_1, \psi_2) \).

**Proposition 4** The group \( \text{Aut}(\mathfrak{so}(4)) \) is generated by \( \text{Int}(\mathfrak{so}(4)) \) and the automorphism \( \zeta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

**Proof:** Let \( M \in \text{Aut}(\mathfrak{so}(4)) \). We show that there exist \( N_1, \ldots, N_k \in \text{Int}(\mathfrak{so}(4)) \cup \zeta \) such that \( N_1 \cdots N_k = M = 1 \). Write

\[
M = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}.
\]

There exists a rotation \( \psi_1 \in \text{SO}(3) \) such that \( \psi_1 \cdot [a_1 \ b_1 \ c_1]^T = [a_1' \ 0 \ 0]^T \) with \( a_1' \geq 0 \). There exists another rotation \( \psi_1' \), preserving \( [a_1' \ 0 \ 0]^T \), such that \( \psi_1' \cdot [a_2 \ b_2 \ c_2]^T = [a_2' \ b_2' \ 0]^T \) with \( b_2' \geq 0 \). Therefore the top-left block of \( (\psi_1', 1) \cdot M \) is

\[
\begin{bmatrix} a_1' & a_2' & a_3' \\ 0 & b_2' & b_3' \\ 0 & 0 & c_3' \end{bmatrix}.
\]

Similarly, by the application of an automorphism (preserving the top-left block), the entries \( e_4, f_4, f_5 \) can be made zero and the entries \( d_4, e_5 \) can be made non-negative. Thus there exists an automorphism \( N_1 \) such
that $N_1 \cdot M = M'$, where

$$M' = \begin{bmatrix}
    a_1' & a_2' & a_3' & a_4' & a_5' & a_6' \\
    0 & b_2' & b_3' & b_4' & b_5' & b_6' \\
    0 & 0 & c_3' & c_4' & c_5' & c_6' \\
    d_1' & d_2' & d_3' & d_4' & d_5' & d_6' \\
    e_1' & e_2' & e_3' & e_4' & e_5' & e_6' \\
    f_1' & f_2' & f_3' & 0 & 0 & f_6'
\end{bmatrix}$$

and $a_1', b_2', d_4', e_5' \geq 0$.

As $M'$ is a Lie algebra automorphism, $M' \cdot [E_i, E_j] = [M' \cdot E_i, M' \cdot E_j]$, $i, j = 1, \ldots, 6$. Hence $a_2', a_3', b_2', d_5', d_6', e_6'$ are all zero. Also $a_1' = b_2'c_2'$, $b_2' = a_1'c_1'$, $c_3' = a_1'b_2'$, $d_4' = e_5'f_6'$, $e_5' = d_4'f_6'$, and $f_6' = d_4'e_5'$. Consequently, the diagonal entries are either all zero or all one.

Suppose the diagonal entries are all one. Then

$$M' = \begin{bmatrix}
    1 & 0 & 0 & a_4' & a_5' & a_6' \\
    0 & 1 & 0 & b_4' & b_5' & b_6' \\
    0 & 0 & 1 & c_4' & c_5' & c_6' \\
    d_1' & d_2' & d_3' & 1 & 0 & 0 \\
    e_1' & e_2' & e_3' & 0 & 1 & 0 \\
    f_1' & f_2' & f_3' & 0 & 0 & 1
\end{bmatrix}$$

Again, we impose the condition that $M'$ preserves the Lie bracket. Simple calculations show that $M' = 1$.

Suppose the diagonal entries of $M'$ are all zero. Then

$$\zeta \cdot M' = \begin{bmatrix}
    d_1' & d_2' & d_3' & 0 & 0 & 0 \\
    e_1' & e_2' & e_3' & 0 & 0 & 0 \\
    f_1' & f_2' & f_3' & 0 & 0 & 0 \\
    0 & 0 & 0 & a_1' & a_2' & a_3' \\
    0 & 0 & 0 & b_1' & b_2' & b_3' \\
    0 & 0 & 0 & c_1' & c_2' & c_3'
\end{bmatrix}.$$  

A similar argument shows that there exists an automorphism $N_2$ such that $N_2 \cdot \zeta \cdot M' = 1$. \hfill \Box

4 Equivalence

We shall consider $\mathcal{L}$-equivalence of homogeneous systems on $\text{SO}(4)$. Such a system $\Sigma = (\text{SO}(4), \Xi)$ is uniquely determined by its parametrization map $\Xi(1, \cdot) : \mathbb{R}^6 \to \mathfrak{so}(4)$. Let $A = \sum_{i=1}^{6} a_i E_i$ and $B = \sum_{i=1}^{6} b_i E_i$. Any automorphism of $\mathfrak{so}(4)$ preserves the dot product $A \cdot B = \sum_{i=1}^{6} a_i b_i$. Let $\Gamma^\perp$ denote the orthogonal complement of a subspace $\Gamma \subset \mathfrak{so}(4)$. The following result is easy to prove.

**Proposition 5** Let $\Gamma, \tilde{\Gamma}$ be subspaces of $\mathfrak{so}(4)$ and let $\psi \in \text{Aut}(\mathfrak{so}(4))$. Then

$$\psi \cdot \Gamma = \tilde{\Gamma} \iff \psi \cdot \Gamma^\perp = \tilde{\Gamma}^\perp.$$  

For the sake of convenience we shall use $\rho_1(\theta)$, $\rho_2(\theta)$, and $\rho_3(\theta)$ to denote, respectively, the rotations

$$\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & \cos \theta & -\sin \theta & 0 \\
    0 & \sin \theta & \cos \theta & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \quad \begin{bmatrix}
    \cos \theta & 0 & \sin \theta & 0 \\
    0 & 1 & 0 & 0 \\
    -\sin \theta & 0 & \cos \theta & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}.$$  

**Example 6** Let $\Xi(1, u) = u_1 (E_1 + \sqrt{3}E_3 + E_4) + u_2(3E_2 + E_5 + E_6)$. The trace of this system is given by $\Gamma = \langle E_1 + \sqrt{3}E_3 + E_4, 3E_2 + E_5 + E_6 \rangle$. Now $\rho_2(\frac{\pi}{2})$ is an automorphism such that

$$\langle \rho_2(\frac{\pi}{2}), 1 \rangle \cdot \Gamma = \langle \cos \frac{\pi}{2} E_1 - \sin \frac{\pi}{2} E_3 + \sqrt{3} (\sin \frac{\pi}{2} E_1 + \cos \frac{\pi}{2} E_3) + E_4, 3E_2 + E_5 + E_6 \rangle = \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle.$$  

Also, $\langle 1, \rho_1(-\frac{\pi}{2}) \rangle$ is an automorphism such that

$$\langle 1, \rho_1(-\frac{\pi}{2}) \rangle \cdot \langle 2E_1 + E_4, 3E_2 + E_5 + E_6 \rangle = \langle E_1 + \frac{1}{2} E_4, E_2 + \sqrt{2} E_6 \rangle = \tilde{\Gamma}.$$  

Therefore, $\Sigma$ is $\mathcal{L}$-equivalent to $\tilde{\Xi}(1, u) = u_1 (E_1 + \frac{1}{2} E_4) + u_2(E_2 + \sqrt{2} E_6)$.

By proposition 5, it follows that $\psi \cdot \Gamma^\perp = \tilde{\Gamma}^\perp$, where $\psi = \rho_2(\frac{\pi}{2}), \rho_1(-\frac{\pi}{2})$. Hence any system with trace

$$\Gamma^\perp = \langle E_1 - E_4 + E_5 - E_6, 2E_2 + E_4 - \sqrt{3}E_4 - 6E_6, E_3 - \sqrt{3}E_4, E_1 + E_2 - E_4 - 3E_6 \rangle$$  

is $\mathcal{L}$-equivalent to one with trace

$$\tilde{\Gamma}^\perp = \langle E_3, E_6, E_1 - 2E_4, E_2 - \frac{3}{\sqrt{2}} E_6 \rangle.$$  

**Theorem 7** Any one-input system is $\mathcal{L}$-equivalent to a system

$$\Xi_1^{\frac{1}{2}}(1, u) = u_1 E_1$$  

$$\Xi_{2, \alpha}^{\frac{1}{2}}(1, u) = u_1 (E_1 + \alpha E_4)$$  

for some $0 < \alpha \leq 1$.

**Proof:** Any single-input system has trace $\Gamma_1 = \langle A_1 \rangle$, $\Gamma_2 = \langle A_2 \rangle$, or $\Gamma_3 = \langle A_1 + A_2 \rangle$. Here $A_1 = \sum_{i=1}^{3} a_i E_i$ and $A_2 = \sum_{i=4}^{6} a_i E_i$ are nonzero. For
Any five-input system is $\Sigma$-equivalent to a system

$$
\Xi_1^5(1, u) = u_1 E_2 + u_2 E_3 + u_3 E_4 \\
+ u_4 E_5 + u_5 E_6
$$

for some $0 < \alpha \leq 1$.

**Remark 9** Any five-input system has full rank.

**Lemma 10** Any system $\Xi(1, u) = u_1 (E_1 + \gamma_1 E_4 + \alpha E_5) + u_2 (E_2 + \gamma_2 E_5)$, $\gamma_1, \gamma_2, \alpha > 0$ is $\Sigma$-equivalent to a system $\Xi'(1, u) = u_1 (E_1 + \gamma'_1 E_4 + \alpha' E_5) + u_2 (E_2 + \gamma'_2 E_5)$ for some $\gamma'_1, \alpha' > 0$.

**Proof:** It suffices to show that there exists an automorphism $\varphi = (\rho_3(\theta_1), \rho_3(\theta_2))$ such that

$$
\varphi \cdot (E_1 + \gamma_1 E_4 + \alpha E_5) = r_1 (E_1 + \gamma'_1 E_4 + \alpha' E_5) \\
+ r_2 (E_2 + \gamma'_1 E_5)
$$

$$
\varphi \cdot (E_2 + \gamma_2 E_5) = r_3 (E_1 + \gamma'_1 E_4 + \alpha' E_5) \\
+ r_4 (E_2 + \gamma'_1 E_5)
$$

for some $\alpha', \gamma'_1 > 0$, $r_1, r_2, r_3, r_4 \in \mathbb{R}$. (Here $(r_1, r_2)$ and $(r_3, r_4)$ are required to be linearly independent.) This reduces to showing that (the system of equations)

$$
\begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1
\end{bmatrix}
\begin{bmatrix}
r_1 & r_2 \\
r_3 & r_4
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
\alpha \sin \theta_2 + \gamma_1 \cos \theta_2 & \alpha \cos \theta_2 - \gamma_1 \sin \theta_2 \\
\gamma_2 \sin \theta_2 & \gamma_2 \cos \theta_2
\end{bmatrix}
\begin{bmatrix}
\gamma'_1 & r_1 \alpha' + \gamma'_1 \\
\gamma'_1 & r_3 \alpha' + r_4 \gamma'_1
\end{bmatrix}
$$

has a solution.

The values $r_1, r_2, r_3, r_4$ are fixed by the first equation (in terms of $\theta_1$). From the second equation we then get (by equating the first columns)

$$
\begin{bmatrix}
\alpha \sin \theta_2 + \gamma_1 \cos \theta_2 \\
\gamma_2 \sin \theta_2
\end{bmatrix}
= \begin{bmatrix}
\gamma'_1 & r_1 \alpha' + \gamma'_1 \\
\gamma'_1 & r_3 \alpha' + r_4 \gamma'_1
\end{bmatrix}.
$$

For any $\theta_2$ there always exist $\theta_1$ and $\gamma'_1 > 0$ satisfying this equation. Hence $\cos \theta_1 = \frac{1}{\gamma'_1} (\alpha \sin \theta_2 + \gamma_1 \cos \theta_2)$ and $\sin \theta_1 = \frac{1}{\gamma'_1} (\gamma_2 \sin \theta_2)$.

It remains to be shown that

$$
\begin{bmatrix}
\alpha \cos \theta_2 - \gamma_1 \sin \theta_2 \\
\gamma_2 \cos \theta_2
\end{bmatrix}
= \begin{bmatrix}
\gamma'_1 & r_1 \alpha' + \gamma'_1 \\
\gamma'_1 & r_3 \alpha' + r_4 \gamma'_1
\end{bmatrix}
$$

has a solution (for some $\alpha'$ and $\theta_2$). This reduces to

$$
\begin{bmatrix}
\gamma_1 \alpha' - \alpha'_1 & \alpha' & \gamma'_1 (1 - \gamma_2) \\
\gamma'_1 (1 - \gamma_1) & \gamma_2 \alpha' + \alpha'_1
\end{bmatrix}
\begin{bmatrix}
\cos \theta_2 \\
\sin \theta_2
\end{bmatrix} = 0.
$$

For $\alpha' = \sqrt{\alpha^2 + (\gamma'_1 - \gamma_1)^2 (\gamma'_1)}$, the determinant of the matrix is zero. Thus there exists a solution (for some $\theta_2$).

A similar argument yields the following result.

**Lemma 11** Any system $\Xi(1, u) = u_1 (E_1 + \gamma_1 E_4) + u_2 (E_2 + \gamma_2 E_5)$, $\gamma_1, \gamma_2 > 0$ is $\Sigma$-equivalent to a system $\Xi'(1, u) = u_1 (E_1 + \gamma'_1 E_4 + \alpha' E_5) + u_2 (E_2 + \gamma'_1 E_5)$ for some $\gamma'_1 > 0$, $\alpha' \geq 0$.

**Theorem 12** Any two-input system is $\Sigma$-equivalent to a system

$$
\Xi_1^2(1, u) = u_1 E_1 + u_2 E_2 \\
\Xi_2^2(1, u) = u_1 E_1 + u_2 E_4 \\
\Xi_3^2(1, u) = u_1 E_1 + u_2 E_2 + \alpha E_5 \\
\Xi_4^2(1, u) = u_1 (E_1 + \alpha E_4 + \beta E_5) \\
+ u_2 (E_2 + \alpha E_5)
$$

for some $\alpha > 0$, $\beta \geq 0$.

**Proof:** Let $\Xi$ be a two-input system with trace $\Gamma = \langle A_1 + A_2, B_1 + B_2 \rangle$. Here $A_1 = \sum_{i=1}^{3} a_i E_i$, $A_2 = \sum_{i=4}^{6} a_i E_i$, $B_1 = \sum_{i=1}^{3} b_i E_i$, and $B_2 = \sum_{i=4}^{6} b_i E_i$.

Suppose $A_1$, $A_2$, $B_1$, $B_2 \neq 0$, $\langle A_1 \rangle \neq \langle B_1 \rangle$ and $\langle A_2 \rangle \neq \langle B_2 \rangle$. There exists $(\psi_1, 1) \in \text{Int}(\alpha (4))$ such that $(\psi_1, 1) \cdot A_1 = r E_1$, $r = \sqrt{A_1 \circ A_1}$. Thus

$$(\psi_1, 1) \cdot \Gamma' = \langle r E_1 + A_2, b'_1 E_1 + b'_2 E_2 + b'_3 E_3 + B_2 \rangle$$

for some $b'_1, b'_2, b'_3 \in \mathbb{R}$.
Likewise, a somewhat more involved argument yields the following result. The proof is omitted.

**Theorem 14** Any three-input homogeneous system is \( \Sigma \)-equivalent to a system

\[
\Xi_1^{3,\alpha,\beta}(1, u) = u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_2 + \beta E_5)
\]

for some \( \alpha_1 \geq \alpha_2 \geq |\beta| \geq 0 \) and \( 0 \leq \gamma \leq 1 \). Here \( \alpha_1 \neq \alpha_2 \) or \( \alpha_2 \neq |\beta| \).

**Remark 15** There is only one 6-dimensional affine subspace of \( \mathfrak{s}(4) \), namely \( \mathfrak{s}(4) \). Therefore any six-input system is \( \Sigma \)-equivalent to the system

\[
\Xi^6(1, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 + u_5 E_5 + u_6 E_6.
\]

### 5 An illustrative example

Any system

\[
\Xi_\gamma(1, u) = \gamma_1 E_1 + u_1(\gamma_1 E_1 + \gamma_3 E_2 + \gamma_4 E_3)
\]

is \( \Sigma \)-equivalent to the system

\[
\tilde{\Xi}(1, u) = u_1 E_2 + u_2 E_4 + u_3 E_4 + u_4 E_5.
\]

Here \( \gamma_i > 0 \), \( i = 1, \ldots, 9 \). As these systems have full rank, they are in fact detached feedback equivalent.

The automorphism relating the traces of these systems is given by \( \psi = (\psi_1, 1) \), where

\[
\psi_1 = \begin{bmatrix}
\frac{\gamma_1}{\sqrt{\gamma_2^3 + \gamma_3^3}} & 0 & 0 & 0 & 0 \\
\frac{\gamma_4}{\sqrt{\gamma_2^3 + \gamma_3^3}} & \frac{\gamma_5}{\sqrt{\gamma_2^3 + \gamma_3^3}} & 0 & 0 & 0 \\
0 & \frac{\gamma_6}{\sqrt{\gamma_2^3 + \gamma_3^3}} & \frac{\gamma_7}{\sqrt{\gamma_2^3 + \gamma_3^3}} & 0 & 0 \\
0 & 0 & 0 & \frac{\gamma_8}{\sqrt{\gamma_2^3 + \gamma_3^3}} & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

The corresponding feedback transformation \( \varphi \), defined by

\[
\psi \cdot \Xi_\gamma(1, u) = \tilde{\Xi}(1, \varphi(u))
\]

is given by

\[
\varphi : u \mapsto \begin{bmatrix}
\gamma_4 & \gamma_5 & 0 & 0 & 0 \\
0 & \gamma_6 & \gamma_7 & 0 & 0 \\
0 & 0 & 0 & \gamma_8 & 0 \\
\end{bmatrix} u + \begin{bmatrix} 0 \end{bmatrix}.
\]
There exists a group automorphism \( \phi \) such that \( T_1 \phi = \psi \). The map \( \phi \) establishes a one-to-one correspondence between trajectories of \( \Xi_1 \) and \( \bar{\Xi} \). Specifically, \( (g(\cdot), u(\cdot)) \) is a trajectory-control pair of \( \Xi_1 \) if and only if \( (\phi \circ g(\cdot), \varphi \circ u(\cdot)) \) is a trajectory-control pair of \( \bar{\Xi} \). This reduces the study of trajectories of \( \Xi_1 \) to the study of trajectories of \( \bar{\Xi} \). In particular, any \( \phi \)-invariant property (e.g., periodicity) can be investigated on \( \Xi_1 \) rather than on \( \Xi_1 \).

### 6 Concluding remarks

In this paper we obtained an exhaustive list of equivalence representatives for homogeneous systems on \( \text{SO}(4) \). In order to get a classification of systems, it is required to make sure that no two equivalence representatives are equivalent.

It is easy to verify that the systems \( \Xi_1 \) and \( \Xi_{1,2,3} \) are not equivalent (for any \( 0 < \alpha \leq 1 \)). It is also easy to show that the systems \( \Xi_{1,2,3} \) and \( \Xi_{1,2,3} \) \( \alpha \) are \( \mathcal{E} \)-equivalent only if \( \alpha = \alpha' \). Hence theorem 7 and 8 provide a classification of systems.

For the two-input systems such a verification becomes quite involved. Clearly,

\[
\Xi_1(1, u) = u_1 E_1 + u_2 E_2
\]

is not \( \mathcal{E} \)-equivalent to

\[
\Xi_2(1, u) = u_1 E_1 + u_2 E_2
\]

as \( \psi \cdot \langle E_1, E_2 \rangle \subset \langle E_1, E_2, E_3 \rangle \) or \( \psi \cdot \langle E_1, E_2 \rangle \subset \langle E_1, E_2, E_3 \rangle \) for any automorphism \( \psi \). Likewise, it is fairly easy to show that none of \( \Xi_1, \Xi_2, \Xi_3, \alpha \), or \( \Xi_{1,2,3} \) \( \alpha \) are \( \mathcal{E} \)-equivalent to one another. Now suppose \( \Xi_{1,2,3} \) and \( \Xi_{1,2,3} \) \( \alpha \) are \( \mathcal{E} \)-equivalent. Any element of \( \text{SO}(3) \) can be written (in terms of Euler angles) as

\[
R_{\theta_3 \theta_2 \theta_1} = \rho_3(\theta_3) \rho_2(\theta_2) \rho_1(\theta_1).
\]

Hence, any automorphism \( \psi \) can be written as

\[
\psi = (R_{\theta_2 \theta_1}, R_{\theta_2 \theta_1}) \text{ or } \psi = \zeta \circ (R_{\theta_2 \theta_1}, R_{\theta_2 \theta_1}).
\]

The system \( \Xi_{1,2,3} \) \( \alpha \) is \( \mathcal{E} \)-equivalent to \( \Xi_{1,2,3} \) \( \alpha' \) if and only if there exist linearly independent vectors \( (r_1, r_2), (r_3, r_4) \in \mathbb{R}^2 \) such that

\[
\psi \cdot E_1 = r_1 E_1 + r_2 (E_2 + \alpha E_3) \quad \text{and} \quad \psi \cdot (E_2 + \alpha E_3) = r_3 E_1 + r_4 (E_2 + \alpha' E_3)
\]

for some automorphism \( \psi \). Applying an automorphism \( \psi = \zeta \circ (R_{\theta_2 \theta_1}, R_{\theta_2 \theta_1}) \), we immediately get that \( r_1 = r_2 = 0 \), a contradiction. If we apply an automorphism \( \psi = (R_{\theta_2 \theta_1}, R_{\theta_2 \theta_1}) \), then we get

\[
\begin{bmatrix}
\cos \theta_2 \cos \theta_3 & \cos \theta_2 \sin \theta_3 - \cos \theta_1 \sin \theta_3 \\
- \sin \theta_2 & \cos \theta_2 \sin \theta_3 + \sin \theta_1 \sin \theta_3 \\
\end{bmatrix}
\begin{bmatrix}
\cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \\
\cos \theta_2 \sin \theta_1 \\
0 & \alpha \cos \theta_1 \sin \theta_3 - \cos \theta_4 \sin \theta_6 \\
0 & \alpha \cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \\
0 & \alpha \cos \theta_1 \sin \theta_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
\frac{r_1}{r_2} & \frac{r_3}{r_4} \\
0 & 0 \\
r_2 \alpha' & r_4 \alpha' \\
0 & 0 \\
\end{bmatrix}
\]

Therefore \( \sin \theta_2 = 0 \) and \( \cos \theta_2 = \pm 1 \). Hence \( \sin \theta_1 = 0 \) and \( \cos \theta_1 = \pm 1 \). Further simplification yields \( \alpha = \alpha' \). Therefore, \( \Xi_{1,2,3} \) \( \alpha \) and \( \Xi_{1,2,3} \) \( \alpha' \) are \( \mathcal{E} \)-equivalent if \( \alpha = \alpha' \). For \( \Xi_{1,2,3} \) \( \alpha \) such a computation becomes more involved.

A verification for three-input systems becomes fairly complicated. We do not know whether theorems 12 and 14 provide a classification.

### References


