Non-axisymmetric vibrations of stepped cylindrical shells

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Abstract: Vibrations of stepped circular cylindrical shells with cracks are studied. The shell under consideration is subdivided into multiple segments separated by the locations of thickness variations. It is assumed that at the boundary of these segments circumferential surface cracks with are located. The influence of circular cracks with constant depth on the vibration of the shell is prescribed with the aid of a matrix of local flexibility. The latter is related to the coefficient of the stress intensity known in the linear fracture mechanics. Numerical results are obtained for various crack models including those, where the shell with a crack is simulated geometrically as a two-stepped shell with small length of the intermediate section. Shells with various combinations of boundary conditions can be analyzed by the proposed method. The influences of the shell thicknesses, locations of step-wise variations of the thickness and other parameters on the natural frequencies are examined. The results can be used for the approximate evaluation of dynamic parameters of simply supported or clamped cylindrical shells with cracks and flaws.

Key words: vibration, cylindrical shell, crack, elasticity

1 Introduction
A suitable measure for the intensity of the stress concentration is assumed to be the local compliance of a cracked body. It can be related by energy arguments to the strain energy concentration and furthermore to the stress intensity factor. This idea was used for experimental determination of the stress intensity factor and a wealth of results were tabulated for a number of cases (Tada et al. [20]). The macroscopic influence of the local flexibility (or compliance) on the static, dynamic and stability behavior of a cracked structure was recognized by many authors (Irvin, [12]). In early works, particularly in civil engineering, a crack was usually considered as a link (Harrison [11]), or as a portion of the structural member with smaller cross-section. Such approach and an equivalent bending moment at the ends of the reduced section was used for an empirical determination of the change in dynamic response due to the crack.

Vibration analysis was also performed for simple structural members with a single spring (Dimarogonas [6]; Chondros and Dimarogonas, [4]). Moreover, the coupling effects were recognized for bending and longitudinal motions (Dimarogonas and Paipetis [5]). Analytical methods with the stiffness or compliance terms in the set of intermediate conditions were applied to simple structural elements. For more complicated structures finite element techniques are widely used.

2 Summary of the basic equations
Let \( x, \theta, z \) be the cylindrical coordinates for a cylindrical shell of length \( l \), as shown in Fig.1, where \( x \) and \( \theta \) are surface coordinates and \( z \) is the inward normal to the reference surface. The origin of the coordinate system is located on the middle surface of the shell, and the radius of the middle surface is denoted by \( R \).

The cylindrical coordinate system \(( x, \theta, z)\) is considered in order to take advantage from the axial symmetry of the structure, the origin of the reference system is located at the centre of left end of the shell. The shell thickness is \( h(x)=h_j \) for \( x \in (a_j,a_{j+1}) \), where \( j=0,...,n \). Here the quantities \( h_j (j=0,...,n) \) stand for fixed constants. Similarly, \( a_j (j=0,...,n+1) \) are given constants whereas it is reasonable to use notations \( a_0=0, a_{n+1}=l \). In Fig.1 three displacement fields are represented: axial \( u(x, \theta, t) \), circumferential \( v(x, \theta, t) \), and radial displacements \( w(x, \theta, t) \), where \( t \) is time. Assume that the ends of the shell are simply supported.

Fig.1: Circular cylindrical shell: coordinate
A system of displacement equilibrium equations, based on Donnell’s approximations for a cylindrical shell can be presented as [8]

\[
\frac{\partial^2 u_j}{\partial x^2} + \frac{1 - \nu}{2R^2} \frac{\partial^2 u_j}{\partial \theta^2} + \frac{1 + \nu}{2} \frac{\partial^2 v_j}{\partial \theta^2} - \nu \frac{\partial w_j}{\partial \theta} = 0,
\]

\[
\frac{1}{2} \frac{\partial v_j}{\partial x} + \frac{1}{2R} \frac{\partial v_j}{\partial \theta} + R \frac{\partial w_j}{\partial x} + \frac{1}{2R} \frac{\partial w_j}{\partial \theta} = 0,
\]

\[
\frac{\partial u_j}{\partial x} + \frac{1}{12} \frac{\partial}{\partial \theta} \left( \frac{h_j^2}{R} \frac{\partial^2 v_j}{\partial \theta^2} + \frac{1}{2} \frac{\partial^2 w_j}{\partial \theta^2} \right) - \frac{1}{R} \frac{\partial w_j}{\partial \theta} = 0.
\]

In accordance with the simplifications leading to equations (163), the effect of the displacements on the bending and twisting moments must be considered as negligible. Thus, with the notation

\[
D_j = \frac{Eh_j^3}{12(1 - \nu^2)},
\]

the following expressions are obtained for the bending moment \( M_{ij} \), shear force \( Q_{i} \) and membrane force \( N_{i} \):

\[
(M_{ij})_j = -D_j \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\nu}{R} \frac{\partial^2 w_j}{\partial \theta^2} \right),
\]

\[
(Q_{i})_j = -D_j \frac{\partial}{\partial x} \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w_j}{\partial \theta^2} \right),
\]

\[
(N_{i})_j = \frac{Eh_j}{1 - \nu^2} \left( \frac{\partial^2 w_j}{\partial x^2} + \frac{\nu}{R} \frac{\partial^2 w_j}{\partial \theta^2} \right).
\]

Donnell [8] has received from the system of equations (1-3) by using a special function \( \varphi \) a following equation for \( w_j \)

\[
\frac{Eh_j^3}{12(1 - \nu^2)} \nabla^4 w_j + \frac{Eh_j}{12} \frac{\partial^4 w_j}{\partial x^4} = \nu^4 p_j.
\]

Here \( p_j = -p \frac{\partial^2 w_j}{\partial x^2} \) and \( \nabla^4 = (\nabla^2)^2 \), \( \nu^4 = (\nabla^2)^2 \), where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{2R} \frac{\partial^2}{\partial \theta^2} \).

Later the equation (6) has been specified by Donnell [8] as

\[
\frac{Eh_j^3}{12(1 - \nu^2)} \left( \nabla^2 + \frac{1}{2R^2} \left( \frac{\nabla^2}{\nabla^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \right) \right) + \frac{1}{2R^2} \left( \frac{\nabla^2}{\nabla^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \right) \right) \frac{Eh_j}{12} \frac{\partial^4 w_j}{\partial x^2} = \nu^4 p_j.
\]

Let us substitute

\[
w_j(x, \theta, t) = e^{i \omega t} \cos \theta \sin \omega t,
\]

where \( p \) is the number of waves in the circumferential direction; \( \omega \) – the circular natural frequency.

This gives with (6) the characteristic equation

\[
\frac{Eh_j^3}{12(1 - \nu^2)} (r_j^2 - \frac{p^2}{R^2}) + \frac{Eh_j}{R^2} - \rho \nu \omega^2 (r_j^2 - \frac{p^2}{R^2}) = 0.
\]

The exact solution of the eighth order equation (6) is quite complicated (Soedel, [17]). It was shown by Soedel (2004) that in particular cases one can find simple approximations to the exact solution.

We assume here in that the general solution of (6) can be presented in the form

\[
w_j(x, \theta, t) = (A_{0j} \sin \nu x + A_{2j} \cos \nu x + A_{3j} \sinh \nu x + A_{4j} \cosh \nu x)cosp \theta \sin \omega t,
\]

where \( r_j = \frac{1}{R} \sqrt{\frac{a R}{h_j - p^2}}, \quad r_{j} = \frac{1}{R} \sqrt{\frac{a R}{h_j + p^2}},
\]

\[
\omega_{mp} = \sqrt{\frac{E}{\rho 12(1 - \nu^2)}} \frac{h_j}{R} \sqrt{\lambda_m^2 + 12(1 - \nu^2) \frac{R^2}{h_j^2} \left( \lambda_m - \frac{p^2}{\lambda_m^2} \right)^2},
\]

where \( \lambda_m = \kappa_m R / h_j \).

If the natural frequencies are obtained from equation (7) then

\[
\omega_{mp} = \sqrt{\frac{E}{\rho 12(1 - \nu^2)}} \frac{h_j}{R} \sqrt{\lambda_m^2 + 12(1 - \nu^2) \frac{R^2}{h_j^2} \left( \lambda_m - \frac{p^2}{\lambda_m^2} \right)^2}.
\]

Let the vibration frequency \( \omega \) be expressed in terms of a non-dimensional frequency parameter

\[
\Omega = \omega R \sqrt{\frac{\rho}{E}}.
\]

Accounting for this notation the equation (11) can be written as

\[
\Omega_{mp} = \frac{h_j}{R} \sqrt{\lambda_m^2 + 12(1 - \nu^2) \frac{R^2}{h_j^2} \left( \lambda_m - \frac{p^2}{\lambda_m^2} \right)^2}.
\]
From equation (12) one obtains
\[ \Omega_{nm} = \frac{h_j}{R\sqrt{12}} W_m, \]
where
\[ W_m = \frac{\lambda_m^2}{\lambda_m} - \frac{5(\lambda_m - p^2)p^2 + 2p^4}{\lambda_m} + \frac{3(\lambda_m - p^2)p^2 + p^4}{\lambda_m^2} + 12(1-v^2) \frac{R^2 (\lambda_m - p^2)^2}{h_j^2 \lambda_m}, \]

(13)

3 Continuity condition and local flexibility
Let us study a cylindrical shell with a step at the section \( x=a \). Assume that there exists a circumferential surface crack with uniform depth \( c \) at the cross section \( x=a \) in the cylindrical shell.

Let the segments adjacent to the crack have thicknesses \( h_0 \) and \( h_1 \), respectively (Fig.2).

![Fig.2: Stepped circular cylindrical shell with crack.](image)

For the simplicity sake we assume that the crack is a stable circular surface crack. In this study it is assumed that the surface crack is always open. A longitudinal element of unit width of the shell is shown in Fig.3.

The surface crack in the shell can be modeled as a distributed line spring (see X. Zhu et al., [20]). The presence of the crack in the shell will cause the additional local flexibility. The flexibility of the spring is a function of the local dimensions and the elastic properties of the cracked region. If the local stress-strain state in the shell will result in the discontinuity of the generalized displacement at the both sides of crack’s section, then the deformation at the cracked region can be described according to the local compliance (D. Broek, [20]) matrix \( C_{ij} \).

The axial, tangential, radial and angular displacements or the difference of their values at both sides of the crack can be written as
\[ \partial w_j / \partial x = \partial w_{j+1} / \partial x. \]

(14)

The crack induces the discontinuity of the displacement at \( x=a \), and the local compliance matrix is deduced above. So the discontinuity of displacements or the difference of their values at both sides of the crack can be written as

\[
\begin{bmatrix}
  u_j - u_{j+1} \\
  w_j - w_{j+1} \\
  v_j - v_{j+1}
\end{bmatrix} = [C] \begin{bmatrix}
  N_x \\
  M_x \\
  Q_x
\end{bmatrix}.
\]

(15)

In the current case we have a crack of mode I. Assume that the compliance \( C_{22} \) with respect to bending moment is much larger than the elements \( C_{11}, C_{12}, C_{33}, C_{44} \) of the compliance matrix and thus the local compliance with respect to bending moment is dominating element in the local compliance matrix. Therefore from (25) by use Table 1 we can find

\[
\partial w_j / \partial x - \partial w_{j+1} / \partial x = (C_{22})_{j+1} (M_x)_{j+1},
\]

\[ w_j - w_{j+1} = 0. \]

(16)

The function \( F(s_j) \) from is called shape function as it is different for experimental specimens of different shape. Many authors have investigated the problem of determination of the stress intensity factor for various specimens (among others Brown and Srawley, [3]; Freund and Hermann, [9]; Irwin, [12]; Tada, Paris, Irwin, [20]).

In the present study we are resorting to the data of experiments conducted by Brown and Srawley which can be approximated as (see Tada, Paris, Irwin, [20])

\[
F(s_j) = 1.93 - 3.07 s_j + 14.53 s_j^2 - 25.11 s_j^3 + 25.8 s_j^4.
\]

(17)

Thus in the following we shall use the function defined by (17) and take

\[
(C_{22})_{j+1} = \frac{72\pi}{Eh_{j+1}} f(s_{j+1}).
\]

(18)

here
\[
f(s_j) = 1.862 s_j^2 - 3.95 s_j^3 + 16.375 s_j^4 - 37.226 s_j^5 + 76.81 s_j^6 - 126.9 s_j^7 + 172.5 s_j^8 - 143.97 s_j^9 + 66.56 s_j^{10}.
\]

(19)
4 Boundary conditions

Let us consider the case of a cylindrical shell simply supported at the ends. Such a hinged edge is not able to transmit a moment \( M_e \), needed to enforce the condition \( \frac{\partial w}{\partial x} = 0 \). Assuming also that there is no edge resistance in the direction \( x \), we arrive at the boundary conditions

\[
\begin{align*}
v &= 0, & N_e &= 0 \quad (20) \\
w &= 0, & M_e &= 0 \quad (21)
\end{align*}
\]

at both ends.

From boundary conditions it follows that at the point \( x=0 \):

\[
A_{2n} = A_{4n} = 0
\]

and at the point \( x=l \):

\[
A_{3n} \sin r l + A_{4n} \cos r l = 0,
\]

\[
A_{3n} \sinh r l + A_{4n} \cosh r l = 0
\]

(23)

5 System of recursive equations

For a circular cylindrical shell of constant thickness with simply supported ends from conditions (22-23) and from the general solution \( w_j \) in the form (8) we have

\[
sin r l = 0.
\]

This results in

\[
r l = \pi m; \quad (m=0,1,\ldots)
\]

and

\[
\lambda_{np} = (\pi m R / l)^2 + p^2
\]

(24)

The natural frequencies are then obtained from equations (12) as

\[
\omega_{np} = \frac{E}{\rho 12(1-\nu^2)} \frac{\beta}{R} \sqrt{W_{np}},
\]

(25)

where

\[
W_{np} = (\pi m R / l)^2 + p^2 + 2 \pi m R / l + 2 p^2
\]

\[
	imes \frac{\pi m R / l - 1}{((\pi m R / l)^2 + p^2)^2}.
\]

For this case the specified formula for natural frequencies is

\[
\omega_{np} = \frac{E}{\rho 12(1-\nu^2)} \frac{\beta}{R} \sqrt{W_{np}},
\]

(26)

where

\[
W_{np} = ((\pi m R / l)^2 + p^2)^2 - \frac{5(\pi m R / l)^2 p^2 + 2 p^2}{((\pi m R / l)^2 + p^2)^2} + \frac{3((\pi m R / l)^2 + p^2)^2 + 12(1-\nu^2)}{((\pi m R / l)^2 + p^2)^2} \left( \frac{\pi m R / l}{l} \right)^4.
\]

Let us consider first the shell without any crack. In the case of the shell with unique step continuity conditions for \( w \) at \( x=a \) can be written as

\[
h_j(a,\theta,t) - w_j(a,\theta,t) = 0,
\]

\[
D_j(\frac{\partial^2 w_j(a,\theta,t)}{\partial x^2} + \frac{\nu}{R} \frac{\partial^2 w_j(a,\theta,t)}{\partial \theta^2}) = D_0(\frac{\partial^2 w_j(a,\theta,t)}{\partial x^2} + \frac{\nu}{R} \frac{\partial^2 w_j(a,\theta,t)}{\partial \theta^2}),
\]

\[
D_j(\frac{\partial^2 w_j(a,\theta,t)}{\partial x^3} + \frac{\nu}{R} \frac{\partial^2 w_j(a,\theta,t)}{\partial \theta \partial x^2}) = D_0(\frac{\partial^2 w_j(a,\theta,t)}{\partial x^3} + \frac{\nu}{R} \frac{\partial^2 w_j(a,\theta,t)}{\partial \theta \partial x^2}).
\]

Adding the appropriate boundary conditions to the system (27) and making use of (8) one obtains a linear homogeneous system of equations with respect to unknowns \( A_{ij} \) \((i=1,2,3,4; j=0,1)\). This system has a non-trivial solution if its determinant vanishes. Equalizing the determinant to zero yields the characteristic equation which will be solved numerically.

6 Numerical results

Numerical analyses for simply supported shells are carried out in the case of the shell with

\[
l=0.2m; \quad R=0.2m; \quad h=20; \quad \rho=7850kg/m^3; \quad \nu=0.3; \quad E=2.1\times10^{11}N/m^2.
\]

Results of calculations are presented in Fig. 5 and Tables 1-3.

Table 1 shows natural frequencies for the case of a simply supported cylindrical shell. Present methods 1 and 2 are based on equations (25) and (26), respectively. The quantity \( p \) is the number of...
semiwaves in the circumferential direction. It is seen from Table 1 that the natural frequencies corresponding to the current analytical methods are in good agreement with the natural frequencies obtained by Pellicano [15].

Values of the frequency parameter \( \Omega \) for simply supported long shells are presented in Table 2 for \( h/R=0.05 \). Here the results obtained by Markus [14], and Zhang [18] and by the present method are accommodated in different columns.

It can be observed that the present solutions based on Donnell shell theory are in close agreement with the exact 3D solutions by Markus [14], and with solutions of Zhang [18]. Numerical analyses for simply supported shells with stepped thicknesses and cracks are carried out in the case when: \( h_1=0.009m; l=1.2m; R=0.12m; a/l=0.5; \gamma=h_1/h_0; \nu=0.3 \) whereas the number of steps \( n=1 \). The results of calculations are presented in Tables 3, 4.

The influence of the crack \( c/h \) on the fundamental frequency parameters of simply supported circular cylindrical shells with one thickness variation is depicted in Fig. 5.

Fig. 5 corresponds to the case when the number of semiwaves in the circumferential direction equals to unity whereas. Fig. 5 is associated with the case of four semiwaves. It can be seen from Fig. 5 that the case of deeper cracks corresponds to lower values of the parameter \( \Omega \). On the other hand, the frequency parameter \( \Omega \) is higher for a stepped shell in comparison to that of the appropriate shell of constant thickness.

Let the shell with a crack in \( a/l=0.5 \) be simulated geometrically as a two-stepped shell with small length of the intermediate section as shown in Fig. 4. From Table 3 we can see that values of frequency parameters \( \Omega \) for a simply supported cylindrical shell with a crack corresponding to different models, coincide with sufficient accuracy. Results are especially close each other in the cases of small values of the depth of the crack.

Table 1: Comparison of natural frequencies for a simply supported shell.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural frequencies (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m/p )</td>
<td>Present method 1</td>
</tr>
<tr>
<td>1/5</td>
<td>739.25</td>
</tr>
<tr>
<td>1/6</td>
<td>564.15</td>
</tr>
<tr>
<td>1/7</td>
<td>493.70</td>
</tr>
<tr>
<td>1/8</td>
<td>498.54</td>
</tr>
<tr>
<td>1/9</td>
<td>555.29</td>
</tr>
<tr>
<td>1/10</td>
<td>645.97</td>
</tr>
<tr>
<td>1/11</td>
<td>759.85</td>
</tr>
<tr>
<td>1/12</td>
<td>891.41</td>
</tr>
<tr>
<td>2/10</td>
<td>977.77</td>
</tr>
<tr>
<td>2/11</td>
<td>992.84</td>
</tr>
</tbody>
</table>

Table 2: Comparison of frequency parameters \( \Omega \) for a simply supported cylindrical shell (\( \nu=0.3; l/R=20; m=1 \)).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( h/R=0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markus</td>
<td>Zhang, Xiang</td>
</tr>
<tr>
<td>1</td>
<td>0.016106</td>
</tr>
<tr>
<td>2</td>
<td>0.039233</td>
</tr>
<tr>
<td>3</td>
<td>0.109477</td>
</tr>
<tr>
<td>4</td>
<td>0.209008</td>
</tr>
</tbody>
</table>

Table 3: Frequency parameters \( \Omega \) for a simply supported cylindrical shell with crack (crack model I and II), the case \( p=2; m=1, a/l=0.5 \).

<table>
<thead>
<tr>
<th>crack model I</th>
<th>crack model II</th>
<th>crack model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>( \Omega ) (( \Delta=0.003 ))</td>
<td>( \Omega ) (( \Delta=0.01 ))</td>
</tr>
<tr>
<td>0</td>
<td>0.0854684</td>
<td>0.085409</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0849369</td>
<td>0.0851978</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0838321</td>
<td>0.0849066</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0825535</td>
<td>0.084491</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0815348</td>
<td>0.0838742</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0811993</td>
<td>0.0829309</td>
</tr>
</tbody>
</table>

Fig. 5: Frequency parameters \( \Omega \) for simply supported shells with one-step thickness variation and crack, the case \( p=4; m=1 \).
7 Concluding remarks
Non-axisymmetric vibrations of circular cylindrical shells of piece wise constant thickness are studied. The shells under consideration have circular flaws of constant depth located at the re-entrant corners of steps. These flaws are considered as stationary surface cracks, no attention is paid to the extension of cracks during vibrations. Making use of the simplified system of equilibrium equations suggested by Donnell the influence of the crack parameters on the vibration characteristics is calculated. Calculations have been implemented for shells with various edge conditions and with different numbers of steps of the thickness. Calculations carried out showed that the crack location and its dimensions have strong influence on the natural frequency of vibrations. Results of calculations showed that when the crack depth increases then the frequency of natural vibrations decreases.

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