Revision through Belief Merging under Constraints

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Abstract: Belief revision studies strategies for retracting information in order to maintain consistency when the addition of new evidence to a belief base makes it inconsistent. Such new evidence is usually in the form of a propositional formula which must be preserved after the revision. An ordering of the sentences in the belief base is used to determine priorities among sentences so that those with lower priority can be identified and retracted. This ordering can be difficult to generate and maintain. To address this issue, in this paper we study how to generate an ordering of belief base sentences through a belief merging operator. Model-based belief merging defines the beliefs of a group; it merges all of the belief bases (profile), which may possibly be inconsistent, into a collectively consistent one. Merging operators obtain a consistent belief base from the set of worlds with the help of a distance measure on worlds and an aggregation function over distances. The closest worlds to the belief profile returned as the result of the operator. We extend a belief merging operator found in the literature in order to consider constraints. Our strategy for belief revision considers the new evidence as a constraint and applies the extended merging operator in order to obtain the revised belief base. We have chosen an operator that has already been implemented and we extend the system in order to consider constraints and analyze several properties for this operator.

Key–Words: Artificial intelligence, knowledge modelling, decision support systems, belief revision, operators, mathematical logic

1 Introduction

Belief revision is a framework that characterizes the process of belief change in which an agent revises its beliefs when new evidence is received. Logic-based belief revision has been extensively studied [7, 6, 11]. An agent’s beliefs are usually represented as a belief base and with the help of some order-based strategies the new piece of information is integrated to the belief base to reach a new consistent revised belief base.

Classical belief revision always yields to new information and thus revises the current beliefs to accommodate new evidence. Most studies of belief revision are based on the AGM (Alchourron, Gardenfors & Makinson) postulates [1] which capture this notion of priority and describe the minimal properties a revision process should have. The AGM postulates formulated in the propositional setting in [7], denoted as R1-R6, characterize the behaviour that a revision operator should comply with. For example, R1 called the success postulate, captures the priority of new evidence over the belief base, it requires that the revision result of a belief base \( \psi \) by a proposition \( \mu \) (new information) should always maintain \( \mu \) being believed.

On the other hand, belief merging studies strategies for combining information contained in a set of, possibly inconsistent, belief bases (profile) obtained from different sources in order to produce a single consistent belief base. In an extended framework, called belief merging under integrity constraints, a set of formulae representing the constraints must be respected.

Several merging operators have been defined and characterized in a logical way. Among them, model-based merging operators [10, 8, 15, 9] obtain a belief base from a set of worlds with the help of a distance measure on worlds and an aggregation function over distances. The closest worlds to the input belief profile are the result of the operator. This framework has a good level of generality, due to the variety of distance functions that may be chosen, however, in the literature almost every proposed operator uses classical Dalal distance. Implementation of Dalal based frameworks faces a big issue: the computation of the models of every base in the profile, which could be very expensive.

An alternative distance has been defined in [3, 14], this distance is based on the notion of Partial
Satisfiability (PS). A merging operator based on PS-distance is also provided. This operator does not support the inclusion of integrity constraints. However, as showed by [8], there is a straightforward manner to introduce integrity constraints. The result of the merging has to fulfill a formula representing the integrity constraints\^1 $\mu$, in this way we can restrain the search of worlds to the ones that satisfy $\mu$ and chose the closest to the input belief profile. The PS-operator has been implemented avoiding the computation of profile models. Moreover, this operator can deal with inconsistent bases which have no models, this is an advantage over Dalal based operators that operate only over the profile’s models.

In this paper, we propose modeling revision process through a PS based merging operator. We extend the PS approach in order to consider constraints. Then we consider the new evidence as a constraint and apply a belief PS-operator in order to obtain the revised belief base. We provide the extension of operator’s implementation and analyze several properties for this operator too.

The rest of the paper is organized as follows. After providing some technical preliminaries and stating the characterization of revision process, in Section 3 we extend the PS-operator and its implementation in order to consider integrity constraints. Section 4 introduces our approach and shows the satisfaction of postulates. Finally, we conclude by outlining some future work.

2 Preliminaries

We consider a language $\mathcal{L}$ of propositional logic using a finite ordered set of symbols $P := \{p_1, p_2, ..., p_n\}$. A belief base/theory $K$ is a finite set of propositional formulae of $\mathcal{L}$ representing the beliefs from a source; we identify $K$ with the conjunction of its elements. A literal $l$ is an atom or the negation of an atom.

The set of models of the language is denoted by $\mathcal{W}$, its elements will be denoted by vectors of the form $(w(p_1), ..., w(p_n))$, where $w(p_i) = 1$ or $w(p_i) = 0$ for $i = 1, ..., n$ and the set of models of a formula $\phi$ is denoted by $\text{mod}(\phi)$. $K$ is consistent iff there exists model of $K$. If $\phi$ is a propositional formula or a set of propositional formulae then $\mathcal{P}(\phi)$ denotes the set of atoms appearing in $\phi$. $|P|$ denotes the cardinality of set $P$.

A belief profile $E = \{K_1, ..., K_m\}$ is a multiset (bag) of $m$ belief bases.

\^1For convenience we use the same symbol $\mu$ for representing the new evidence in belief revision and the integrity constraints in belief merging contexts.

Let $\leq_\psi$ a relation over worlds; $x =_\psi y$ is a notation for $x \leq_\psi y$ and $y \leq_\psi x$, and $x <_\psi y$ is a notation for $x \leq_\psi y$ and $y \not\leq_\psi x$.

2.1 Belief Revision Characterization

In [1] eight postulates have been proposed to characterize the process of belief revision, which are known as the AGM Postulates. Assuming a proposition setting, in [7] this characterization is rephrased yielding the following $R_1$-$R_6$ postulates, where $\psi$, $\psi_1$ and $\psi_2$ are belief theories to be revised and $\mu$, $\mu_1$ and $\mu_2$ are new evidence:

$R_1$. $\psi \circ \mu$ implies $\mu$.

$R_2$. If $\psi \land \mu$ is satisfiable, then $\psi \circ \mu \equiv \psi \land \mu$.

$R_3$. If $\mu$ is satisfiable, then $\psi \circ \mu$ is also satisfiable.

$R_4$. If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 \circ \mu_1 \equiv \psi_2 \circ \mu_2$.

$R_5$. $(\psi \circ \mu_1) \land \mu_2$ implies $\psi \circ (\mu_1 \land \mu_2)$.

$R_6$. If $(\psi \circ \mu) \land \phi$ is satisfiable, then $\psi \circ (\mu \land \phi)$ implies $(\psi \circ \mu) \land \phi$.

The authors have introduced the notion of faithful assignment and provided a representation theorem which shows an equivalence between the six postulates and a revision strategy based on total pre-orders, the formal definitions are as follows:

Definition 1 Let $W$ be the set of all worlds (interpretations) of a propositional language $\mathcal{L}$. A function that maps each sentence $\psi$ in $\mathcal{L}$ to a total pre-order $\leq_\psi$ on worlds $W$ is called a faithful assignment if and only if:

1. $w_1, w_2 \models \psi$ only if $w_1 =_\psi w_2$;

2. $w_1 \models \psi$ and $w_2 \not\models \psi$ only if $w_1 <_\psi w_2$; and

3. $\psi \equiv \phi$ only if $\leq_\psi = \leq_\phi$.

Theorem 2 (Representation Theorem) A revision operator $\circ$ satisfies Postulates $R_1$-$R_6$, if there exists a faithful assignment that maps each sentence $\psi$ into a total pre-order $\leq_\psi$ such that:

\[\text{mod}(\psi \circ \mu) = \min(\text{mod}(\mu), \leq_\psi)\]
3 \( PS-Merge \)

We suppose that the integrity constraints do not contradict each other. Moreover, without loss of generality we only consider constraints represented by a unique formula \( \mu \). If there are \( n \) constraints \( \mu_1, \mu_2, \ldots, \mu_n \) we shall represent them by the conjunction of the constraints, i.e. we shall consider only the belief merging case under one constraint \( \mu = \mu_1 \land \mu_2 \land \ldots \land \mu_n \).

3.1 \( PS-Merge \)

Without loss of generality we consider only normalized languages so that each belief base is taken as the disjunctive normal form (DNF) of the conjunction of its elements or the conjunctive normal form (CNF) of the conjunction of its elements.

The notion of partial satisfiability is a generalization of satisfiability, in which the valuation function \( w : L \rightarrow \{0,1\} \) is extended to \( w_{ps} : L \rightarrow [0,1] \), i.e. the range in a PS-valuation could be any number between 0 and 1. Instead of indicating satisfiability with a boolean value, partial satisfaction yields a number representing the degree of satisfaction of a formula. If a formula is unsatisfied, its partial satisfiability is 0. If a formula is satisfiable completely, its partial satisfiability is 1. The partial satisfiability of any other case is between these two values. The authors have proposed two definitions of partial satisfiability: one considers only formulae in DNF and the other, called normal partial satisfiability, considers both forms DNF and CNF. Even when there is a small difference in the valuation of conjunctions, in this paper we consider only normal partial satisfiability, so when we refer partial satisfiability we refer the case of normal partial satisfiability. The difference in the valuation of conjunctions considers a degree of satisfiability when the formula \( K \) is unsatisfied and does not contain all the atoms of the language \( |P(K)| < |L| \), i.e. when the agent is not satisfied at all in its own formula, the partial satisfiability considers a small degree of satisfaction for the atoms not appearing in the formula representing its beliefs.

**Definition 3 (Normal Partial Satisfiability)** Let \( K \in L(P) \) in DNF or CNF, \( w \in W \), the Normal Partial Satisfiability of \( K \) for \( w \), denoted as \( w_{ps}(K) \), is defined as follows:

- If \( K \) is a literal i.e. \( K \in P \) or \( K := \neg p \)
  \[ w_{ps}(K) = w(K) \]
- If \( K := D_1 \lor \ldots \lor D_n \)
  \[ w_{ps}(K) = max \{ w_{ps}(D_1), \ldots, w_{ps}(D_n) \} \]

- If \( K := C_1 \land \ldots \land C_n \)
  \[ w_{ps}(K) = \frac{\sum_{i=1}^{n} w_{ps}(C_i)}{n} \]

**Example 4** The Partial Satisfiability of the belief base \( K = (\neg a \land \neg c) \lor (b \land \neg c) \) given \( P = \{a, b, c\} \) and \( w = (1,1,1) \) is
\[ w_{ps}(K) = \max \left\{ \frac{w(\neg a) + w(\neg c)}{2}, \frac{w(b) + w(\neg c)}{2} \right\} = \frac{1}{2}. \]

**Example 5** The Partial Satisfiability of the belief base \( K = (a \lor b) \land (a \lor c) \) given \( P = \{a, b, c\} \) and \( w = (0,0,1) \) is
\[ w_{ps}(K) = \max \left\{ \frac{w(a) + w(b) + w(c)}{2}, \frac{w(a) + w(c)}{2} \right\} = \frac{1}{2}. \]

**Example 6** Notice that we can find the Partial Satisfiability of inconsistent belief bases such as \( K = a \land (\neg a \lor b) \land \neg b \), given \( P = \{a, b\} \) and \( w = (1,0) \), it is
\[ w_{ps}(K) = \frac{w(a) + \max \{w(\neg a), w(b)\} + w(\neg b)}{3} = \frac{2}{3}. \]

This intuitively means that two out of three conjuncts are satisfied.

In classical model-based belief merging, the process of merging a profile \( \Delta(E) \) defines three distances: a distance from a world to another one \( d(w_1, w_2) \), a distance from a world to a belief base \( d(w, K) \) based on \( d(w_1, w_2) \) and a distance from a world to a profile \( d(w, E) \) based on \( d(w, K) \). The latter distance allows us to define a pre-order. The closest worlds to the profile are the models of the merging process. Partial satisfiability merging is quite similar. The process defines a distance from a world \( w \) to a base \( K \) directly without help of distances between worlds as follows:
\[ d(w, K) = w_{ps}(K), \]
then a distance from a world \( w \) to a profile \( E \) is defined as follows:
\[ d(w, E) = \sum_{K \in E} d(w, K), \]
i.e. the distance from a world to a profile is the sum over all the partial satisfiability of every belief base. Finally, a pre-order for a profile \( E \) is defined as follows:
\[ w_1 \leq_E w_2 \iff d(w_1, E) \geq d(w_2, E). \]

The models of the merging are the worlds that are closest to the profile.
\[ mod(\Delta(E)) = \min(W, \leq_E) \]
The merging process can be extended straightforward when a set of constraints $\mu$ is imposed. To assure that the result of the merging will satisfy the integrity constraints, we can restrict the search to the constraints models $\text{mod}(\mu)$ as follows:

$$\text{mod}(\Delta_\mu(E)) = \min(\text{mod}(\mu), \leq_E).$$

3.2 Algorithm $\text{PS}_\mu$ - Merge

We can easily modify the algorithms proposed in [4, 14] to consider constraints. For example, the algorithm presented in [14] is modified in order to consider constraint as shown in Algorithm 1. The algorithm now considers $\mu$ as an input, as follows:

- $V$ : Number of variables of profile $E$
- $B$ : Number of bases of $E$
- $C$ : Vector of number of conjuncts of each base in $E$
- $L$ : Matrix of occurrences of literals by each conjunct of each base in $E$
- $M$ : Number of conjuncts of constraints $\mu$
- $L_M$ : Matrix of occurrences of literals by each conjunct of constraints $\mu$

Example 7 Consider the base of Example 5, with $E = \{K\} = \{(a \lor b) \land (a \lor c)\}$, and $\mu = \neg a \land \neg b$ then the input data accepted by the Algorithm 1 are as follows:

- $V = 3$
- $B = 1$
- $C = (2)\begin{array}{cccc}a & \neg a & b & \neg b & c & \neg c \end{array}$
- $L = \begin{pmatrix}1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
- $M = 1$
- $L_M = \begin{pmatrix}0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$

The algorithm has been implemented in NU Octave 3.6.2 and was tested with all the examples presented in this article with an average case performance of 0.3 sec, and some other cases with a greater number

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**Algorithm 1: $\text{PS}_\mu$-Merge**

**Data:**
- $V$ : Number of variables of $E$
- $B$ : Number of bases of $E$
- $C$ : Vector of number of conjuncts of each base in $E$
- $L$ : Matrix of occurrences of literals by each conjunct of each base in $E$
- $M$ : Number of conjuncts of constraints $\mu$
- $L_M$ : Matrix of occurrences of literals by each conjunct of constraints $\mu$

**Result:**
- $\text{Solution}_\text{Set}$ : The set of models in $\text{PS}_\mu\text{-Merge}(E)$

**begin**

$$\text{Solution} \leftarrow \emptyset$$
$$\text{Max-Sum} \leftarrow 0$$
$$W \leftarrow \text{Matrix whose rows are the models of } \mu$$

**for** $s = 1 \ldots B$ **do**

$$IC_s \leftarrow \sum_{k=1}^{s-1} C_k + 1$$

**for** $i = 1 \ldots \text{rows}(W)$ **do**

$$\text{Sum} \leftarrow 0$$

**for** $s = 1 \ldots B$ **do**

$$\text{ps-conjunct} \leftarrow 0$$

**for** $c = IC_s \ldots IC_s + C_s$ **do**

$$\text{satisfied} \leftarrow 0$$

**for** $j = 1 \ldots V$ **do**

- if $W_{i,j} = 1$ then
  $$\text{satisfied} \leftarrow L_{c,2j-1}$$
- if $W_{i,j} = 0$ then
  $$\text{satisfied} \leftarrow L_{c,2j}$$

$$\text{ps-conjunct} \leftarrow \text{ps-conjunct} + \text{satisfied}$$

$$\text{PS} \leftarrow \text{ps-conjunct}$$

$$\text{Sum} \leftarrow \text{Sum} + \text{PS}$$

**if** $\text{Sum} > \text{Max-Sum}$ **then**

$$\text{Solution} \leftarrow \{i\}$$

$$\text{Max-Sum} \leftarrow \text{Sum}$$

**else if** $\text{Sum} = \text{Max-Sum}$ **then**

$$\text{Solution} \leftarrow \text{Solution} \cup \{i\}$$

**Solution}_\text{Set} = \{i\text{-th-row of } W \mid i \in \text{Solution} \}$$

**end**
of variables for example 15 variables with 25 bases
and a total of 140 conjuncts, with an average case
performance of 1600 sec. The results indicate that
problems of modest size can be treated using common
hardware and short computation times. The complex-
ity is polynomial.

4 Revision through merging

Now, for adapting PS based framework to a belief re-
vision context it is enough to consider revision as a
particular case of merging under constraints, where
the profile $E$ is a singleton $E = K$ and the constrains
$\mu$ represent the new information. Thus, to revise a
belief base $K$ with new information $\mu$ can be defined
through merging as follows:

$$mod(K \circ \mu) = mod(\Delta_\mu(K))$$

Given the profile is a singleton, the sum of partial
satisfiability of the belief base is not necessary, i.e.
$\forall K' \in E \sum_{K \in E} d(w, K) = d(w, K')$, so we can re-
define the pre-order in terms of distance from a world to
a base as follows:

$$w_1 \leq_K w_2 \iff d(w_1, K) \geq d(w_2, K),$$

which in terms of implementation will reduce the
complexity.

Here, we show the functioning of the PS-operator
on some examples:

Example 8 In [2] the following example is presented:
$K = \{a \lor b, a \lor c\}$ and $\mu = \neg a \land \neg b$. From now we
suppose that atoms are ordered alphabetically. The
models of the revision must be found on the models of
$\mu$: $(0, 0, 1)$ and $(0, 0, 0)$. As we can see in Table 1 the
model of $\Delta_\mu(K)$ is $(0, 0, 1)$.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$d(w, K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 1)$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$(0, 0, 0)$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: $\Delta_\mu(K)$

Example 9 In [2] we can find the following example too: $K = \{a \lor c, \neg b \lor d, \neg a \lor b\}$ and $\mu = \neg c \land \neg d$.
As we can see in Table 2 the models of $\Delta_\mu(K)$ are
$(1, 1, 0, 0), (1, 0, 0, 0)$, and $(0, 0, 0, 0)$.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$d(w, K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 0, 0)$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$(0, 1, 0, 0)$</td>
<td>0</td>
</tr>
<tr>
<td>$(1, 0, 0, 0)$</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

Table 2: $\Delta_\mu(K)$

4.1 Satisfaction of postulates

We show now that the operator satisfies, under certain
restrictions, postulates $R_1$-$R_6$. We use the representa-
tion theorem. Clearly the $\leq_\psi$ is a total pre-order on
the worlds given its definition is based on greater than
or equal to relation over real numbers.

For the first statement $w_1, w_2 \models \psi$ only if $w_1 =_\psi w_2$, suppose that $w_1, w_2 \models \psi$ holds, then the partial
satisfiability of $\psi$ for $w_1$ and $w_2$ is 1 i.e. $d(w_1, \psi) = 1$
and $d(w_2, \psi) = 1$ which means $w_1 =_\psi w_2$.

For the second statement suppose that $w_1 \models \psi$ and
$w_2 \not\models \psi$ hold, then the partial satisfiability of $\psi$
for $w_1$ is 1 and the partial satisfiability of $\psi$ for $w_2$ is
less that 1, i.e. $d(w_1, \psi) > d(w_2, \psi)$, which means
that $w_1 <_\psi w_2$.

We can prove the last statement solely when the
formulae have been compiled to the disjunction of
all its prime implicants (a particular case of DNF). If
$\psi \equiv \phi$, let $\psi'$ and $\phi'$ be the the disjunction of prime
implicants of $\psi$ and $\phi$, respectively, then we have
$\leq_\psi = \leq_{\phi'}$ because prime forms satisfy the uniqueness
property (up to the order of the terms and of the literals
that occur within them). The representation in prime
implicants is unique in the sense that, given a set $L$ of
propositional symbols, every proposition built up with
symbols of $L$ has exactly one representation in prime
implicants that represents a whole family of congruent
propositions [12].

While the approach requires compilation to as-
sure independence of syntax, it can deal with in-
consistent belief bases, which is a significant issue rarely
addressed in the literature. Moreover, there is already
some research trying to solve the compilation problem
from a formula to its prime implicants [16, 17].

4.2 Comparing results

Revision using Partial Satisfiability-based merging
yields the same results in some cases compared with
existing techniques such as BHQ [2] or the well
known Dalal [5] approach. Let $K$ be in each case the
belief base and $\mu$ the new evidence consisting of the
belief bases enlisted in Table 3 and let $P$ be cor-
responding set of atoms ordered alphabetically.

As we can see, in the first case Dalal revision
$\neg a \land \neg b$ lost the information concerning $c$, instead the
results of BHQ and PS operators \(-a \land \neg b \land c\) preserve this information. In the second case BHQ revision is the same constraint \(\mu\), instead the result of Dalal operator \((a \leftrightarrow b) \land \neg c \land \neg d\) considers an equivalence between \(a\) and \(b\), and the result of PS-operator \((a \lor b) \land \neg c \land \neg d\) considers a disjunction between \(a\) and \(b\) which assures satisfaction of at least two conjunctions of input \(K = (a \lor c) \land (\neg b \lor d) \land (\neg a \lor b)\).

The last example considers the revision of an inconsistent belief base, Dalal’s revision does not apply in this case, the results of BHQ and PS operators retract \(\neg b\) in order to produce the consistent belief base \(a \land (\neg a \lor b) \land b\).

### 5 Conclusion

Classical belief revision always results in trusting new evidence. Some approaches need extra information, such as priorities between formulae, in order to process the revision. However, in many case this extra information is not available. We propose a new revision method considering “flat” belief bases without extra information. We adapt belief merging in order to carry out belief revision. We use the property of belief merging under constraints which always results in holding the constraints. Instead to create a new operator we consider revision as a particular case of belief merging under constraint. We use the PS operator which has the advantage of having an implementation. We extend the implementation in order to consider the constraints. We take into account of the revision postulates that must be satisfied and prove under certain syntactical restriction that our approach satisfies AGM postulates. The formulae expressed in its prime implicant form satisfy the principle of independence of the syntax. It is worth noticing many revision approaches in the literature use models of operations that, when they face an inconsistent belief base, their function is blocked. However, PS based revision can deal with the inconsistency (see example in Table 3).

Future work considers a deep analyse of PS definition in order to propose a new definition that avoids the restriction concerning prime implicants forms.

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