Portfolio choice: a non parametric Markovian framework

Enrico Angelelli
University of Brescia
Department Economics
and Management
Brescia, Italy
angele@eco.unibs.it

Sergio Ortobelli Lozza
University of Bergamo
Department SAEMQ
24127, Bergamo (Italy) and
VSB-TU of Ostrava
Department of Finance
70121 Ostrava, Czech Republic
sol@unibg.it

Gaetano Iaquinta
Private Research Analyst
20017- Rho (MI)- Italy
gaetano.iaquinta@unibg.it

Abstract: In this paper we argue the large scale dynamic portfolio selection problem when the returns follow a Markov process with heavy tailed distributions. First, we provide a methodology to approximate the portfolios sample paths when the returns follow a Markov process and present heavy tailed distributions. Then, we examine the profitability of some reward-risk strategies applied to large scale portfolio problems. In particular, we compare the ex-post sample paths of the wealth obtained implementing some large scale dynamic portfolio strategies.

Key–Words: dynamic portfolio selection, stable Paretian distributions, Markov chain, market stochastic bounds.

1 Introduction

In this paper we propose a methodology to optimize portfolio value in a choice problem framework using a markovian structure to model the asset portfolio returns. A comparison between different portfolio selection strategies is provided. The proposed methodology is tested in a ex-post analysis and the last crisis period data are used to assess the goodness of the method.

A normal distribution of asset returns is a traditional and basic assumption in many theoretical financial studies. However, many empirical studies rejects the hypothesis that asset returns are normally distributed (see, among others, Rachev and Mittnik (2000), Rachev et al. (2007) and the reference therein). Moreover, many financial events are considered as real witnesses of failure of normal distribution hypothesis in the financial returns (i.e. stock market crash in October 1987, Asian financial crisis in 1997, highly volatile period after September 11, 2001, and the most recent sub-prime mortgage crisis and credit risk crisis (2008-2010)). Therefore a flexibility and statistical reliability in financial model are required to cope with that unrealistic hypothesis. Researchers have spent many efforts to improve methods and propose better models for financial markets. Among the numerous models proposed a fruitful research field appears to be the stable Paretian framework (e.g., Samorodnitsky and Taqqu, 1994) which assumes a financial return distribution more flexible than the traditional one.

Any portfolio dynamic model has to take into account for:

a) Heavy tails and asymmetric shape in returns distribution.

b) A multivariate distribution of underlying asset returns and correlation among asset returns more flexible than the simple Pearson linear correlation.

c) A dynamic portfolio strategy has to be based on the entire sample paths.

In this paper we discuss a portfolio selection model for financial markets based on these three themes with a particular attention to theme c). In order to evaluate and estimate the path dependent portfolio strategies we approximate the return time evolution by using Markovian trees. This approach, originally developed in the option theory (see Cox et al.(1979)), can be efficiently used for portfolio selection problems as shown by Angelelli and Ortobelli (2009). In this framework the evolution of the wealth is derived as a non
parametric process. The Markovian approach allows to compute: the statistical distribution of any contingent claim, the distribution of stopping times or first passage time (see Angelelli and Ortobelli (2009) and Angelelli et al. (2011)), and the joint Markov distributions of risky variables. The portfolio selection strategies based on Markovian trees import several results obtained in option theory: path dependent portfolio selection strategies, arbitrage strategies for hedge funds, and strategies based on stopping times of the random wealth process. In order to account of the dependence structure we use the methodology discussed in Ortobelli et al. (2011) and Angelelli et al. (2011).

In the empirical comparison we analyzes the impact of some proposed portfolio selection strategies applied to US market stock returns data. The ex-post analysis provided is based on two different datasets: the last ten years and the last six months. The use of the two different datasets allows to value the impact of the most recent firms on portfolio selection problems. On these assets the Ortobelli et al. (2011) techniques of dimensionality reduction are applied. Then, the optimal portfolios of different reward-risk strategies are determined. Finally, it is evaluated the impact of considering heavy tails comparing the sample return associated to each state as the geometric average of the extremes of the interval \( [\min X, \max X] \) in the interval \( \min Y, \max Y \) where w.l.o.g. we assume \( z_s(x) > z_s(y) \) for \( s = 1, \ldots, N - 1 \).

In general, the wealth obtained with the portfolio \( x \in S \) at time \( k = 1,2, \ldots \) is a random variable \( W_k(x) \) with a number of possible values increasing as a polynomial of order \( N \) in variable \( k \). In order to keep the complexity of the computation reasonable, we first divide the portfolio selection \( (\min Y, \max Y) \) in \( N \) intervals \( \{a(x),i; a(x),i-1\} \) where \( a(x),i \) (decreasing with index \( i \)) is given by:

\[
a(x),i = \left( \min Y, \max Y \right)_{i/N}, \quad \min Y, \quad i = 0,1, \ldots, N; \quad \text{then, we compute the return associated to each state as the geometric average of the extremes of the interval} \quad (a(x),i; a(x),i-1), \quad \text{that is} \quad z_s(x) = \left( \min Y, \max Y \right)_{(1-2s)/N}, \quad s = 1,2, \ldots, N. \quad \text{As a consequence,} \quad z_s(x) = \left( \min Y, \max Y \right)_{1/N-1} > 1 \quad \text{and the wealth} \quad W_k(x) \quad \text{obtained along a path after} \quad k \quad \text{steps (i.e. at time} \quad k) \quad \text{can}
\]

paid by the asset between \( t \) and \( t+1 \). We distinguish the definition of gross return from the definition of return, i.e., \( z_{t+1} - 1 \) or the alternative definition of log returns \( r_{t+1} = \log z_{t+1} \).
only assume 1 + (N - 1)k distinct values instead of O(k^N). We denote such property as the recombining effect.

Thanks to the recombining effect of the wealth \( W(x) \), the possible values of \( W_k(x) \) up to time \( T \) (i.e., \( k = 1, \ldots, T \)) can be stored in a matrix with \( T \) columns and \( 1 + (N - 1)T \) rows resulting in \( O(NT^2) \) memory space requirement. The transition matrix \( P(x) \) is denoted simply as \( P \). In this paper we omit the reference to the portfolio choice of the portfolio can be tacitly understood, and the transition matrix does not depend on time and it can be simply denoted by \( P(x) \).

In order to simplify the notation, the wealth \( W_k \) is described as \( \{W_k(x)\}_{1 \leq j \leq N} \) valued at time \( k \), and the probability \( P_{i,j,k} \) of the transition process from state \( z_i \) at time \( k+1 \). In this paper we only consider homogeneous Markov chains, so transition matrix does not depend on time and it can be simply denoted by \( P(x) \).

In portfolio literature more than one hundred static reward-risk performance measures have been proposed (see Cogneau and Hübner (2009)). Here, we list the Sharpe static strategy and some OA performance functionals isotonic with choices of non-satiable investors that will be object of the following empirical analysis. For all the OA portfolio strategies we assume that investors have temporal horizon equal to \( T \).

**OA-Sharpe ratio (OA-SR).** The classic version of the Sharpe ratio (see Sharpe (1994)) values the expected excess return for unity of risk (standard deviation). With the OA-Sharpe ratio we value the expected excess final wealth for unity of risk, i.e.,

\[
OA-SR(W_T(x)) = \frac{E(W_T(x) - W_T(r_b))}{\sigma_{W_T(x) - W_T(r_b)}}
\]

where \( W_T(r_b) \) is the final wealth at time \( T \) we obtain investing in the benchmark \( r_b \). In this case we should consider the bivariate Markovian evolution of the vector \( (W_T(x), W_T(r_b)) \) to value the standard deviation \( \sigma_{W_T(x) - W_T(r_b)} \) of \( W_T(x) - W_T(r_b) \). Yet, in the following analyses we assume that the riskless asset is not allowed, thus, the OA-Sharpe Ratio is simply given by \( \frac{E[W_T(x)] - E[W_T(r_b)]}{\sigma_{W_T(x) - W_T(r_b)}} \). When the benchmark \( r_b \) is the risk free rate, the Sharpe ratio is isotonic with non-satiable risk averse preferences. However, using Sharpe type measures we generally don’t take into account the asymptotic behavior of the wealth (except in the case the optimal portfolios are in the domain of attraction of the Gaussian law).

**OA-Asymptotic Sharpe ratio (OA-ASR)**

This performance functional is defined as
$OA-ASR(W_T(x)) = \begin{cases} \frac{\mu_{ln}(W_T(x))}{E(\ln(W_T(x)) - \mu_{ln}(W_T(x)))^{1.01}} & \text{if } \alpha(x) > 1.01 \\ 0 & \text{if } \alpha(x) \leq 1.01 \end{cases}$

where $\mu_{ln}(W_T(x)) = \mu(x)$ is the mean of the stable distribution that better approximates the log final wealth $\ln(W_T(x)) \sim S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x))$ when $\alpha(x) > 1.01$. We assume $OA-ASR(W_T(x)) = 0$ when $\alpha(x) \leq 1.01$ since low indexes of stability imply so heavy tails that the 1.01 moment of the stable distribution is infinite. Observe that when $\alpha = 2$ the final wealth is log normal distributed. Moreover, if $\alpha = 2$ for all the portfolios, the maximization of the OA asymptotic Sharpe ratio is equivalent to the maximization of the Sharpe ratio of the log wealth. As for the Sharpe ratio, this ratio is isotonic with the preferences of non satiable risk averse investors (see Rachev et al. 2008). In order to maximize the $OA-ASR$, we estimate the four stable parameters $(\alpha(x), \sigma(x), \beta(x), \mu(x))$ using the McCulloch’s quantile algorithm (see McCulloch (1986)) and then we compute the 1.01 moment of the centered log wealth.

**OA-Stable stochastic bounds ratio (OA-SSBR)** This performance functional is defined as:

$$OA - SSBR(W_T(x)) = \begin{cases} \frac{E(\Delta_1 - \mu_{\Delta_1})^{1.01}}{E(\Delta_2 - \mu_{\Delta_2})^{1.01}} & \text{if } \alpha_1 \text{ and } \alpha_2 > 2.01 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_1$ and $\alpha_2$ are the indexes of stability respectively of $\Delta_1 = \ln(W_T(x)) - \ln(W_T(\min_i z_i))$, $\Delta_2 = \ln(W_T(\max_i z_i)) - \ln(W_T(x))$, while $W_T(\min_i z_i)$ and $W_T(\max_i z_i)$ are the forecasted wealths at time $T$ obtained respectively by the lower market stochastic bound and the upper market stochastic bound. Moreover $\mu_{\Delta_1} = \mu_1$, $\mu_{\Delta_2} = \mu_2$ are the location parameters of the stable distributions that better approximate respectively $\Delta_1 \sim S_{\alpha_1}(\sigma_1, \beta_1, \mu_1)$ and $\Delta_2 \sim S_{\alpha_2}(\sigma_2, \beta_2, \mu_2)$. In order to determine the distributions of $\ln(W_T(x)) - \ln(W_T(\min_i z_i))$ and $\ln(W_T(\max_i z_i)) - \ln(W_T(x))$ we have to use the evolution of the bivariate Markov processes $(W_i(x), W_i(\min_i z_i))$ and $(W_i(x), W_i(\max_i z_i))$. Recall that, when no short sales are allowed, the upper and the lower market stochastic bounds among $n$ assets with gross returns $z_i$ ($i = 1,...,n$) are respectively given by $\max_{i\leq n} z_i$ and $\min_{i\leq n} z_i$, since $\min_{i\leq n} z_i \leq z(x), \kappa \leq \max_{i\leq n} z_i$ for any time $k$ and for any vector of portfolio weights $x \in S$ (for further generalizations see Ortobelli et al. (2011) and references therein). This ratio expresses the idea that investors want to maximize the distance between the wealth and the lower market stochastic bound, and to minimize the distance between the wealth and the upper market stochastic bound.

**OA-Stable loss ratio (OA-SLoss)** The OA stable loss ratio values the expected asymptotic log wealth for unity of loss. This ratio can be seen as a particular case of the Starr ratio applied to stable distributions (see, among others, Biglova et al. (2004)). Thus, using the asymptotic approximation of log wealth $\ln(W_T(x)) \sim S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x))$ we can easily compute

$$OA - SLoss(W_T(x)) = \begin{cases} \frac{\mu_{ln}(W_T(x))}{-E(\ln(W_T(x))|\ln(W_T(x)) \leq 0)} & \text{if } \alpha(x) > 1 \\ 0 & \text{if } \alpha(x) \leq 1 \end{cases}$$

where $\mu_{ln}(W_T(x)) = \mu(x)$ is the location parameter of the stable distribution that better approximates the final log wealth and $E(\ln(W_T(x))|\ln(W_T(x)) \leq 0)$ is obtained using the Stoyanov et al.’s formula for stable distributions (see Stoyanov et al. (2006)). We assume $OA-SLoss(W_T(x)) = 0$ when $\alpha(x) \leq 1$ since low indexes of stability imply so heavy tails that the first moment of the stable distribution is infinite. The conditional expected loss $-E(X|X \leq 0)$ of an $\alpha$ stable random variable $X \sim S_{\alpha}(\sigma, \beta, \mu)$ is given by: $-E(X|X \leq 0) = \frac{2\Gamma((\alpha-1)/\alpha)}{\pi^{\alpha-2}(\cos(\alpha\theta_0))^{1/\alpha}} \sigma - \mu$ where $\theta_0 = \frac{1}{\alpha} \arctan(\beta \tan(\frac{\pi}{2\alpha}))$ (see Stoyanov et al. (2006)).

In order to reduce the dimensionality of the problem we adopt the techniques developed by Ortobelli et al. (2011) and Angelelli et al. (2011) preselecting no more than 170 assets and then reducing the dimensionality of the preselected assets identifying some common factors to approximate the asset returns.

### 3 An Empirical Comparison Among Portfolio Strategies on the US Stock Market

In this section, we evaluate the impact of the proposed modelization on the the stocks traded on the NYSE and on the NASDAQ in the US stock market. The financial data used in this work are provided by DataStream. In our empirical analysis we use a date set of about two years (500 daily observations) from 15-Sep-2008 till 31-Aug-2010, and assume the following settings:

a) that investors have a temporal horizon of $T = 20$ working days (thus, for each portfolio strategy we...
should optimize the portfolio every 20 working days for a total of 25 optimizations;

b) that investors cannot invest more than ten percent in a single asset (i.e.: \( x_i \in [0, 0.1] \));

c) Markov chains have \( \mathcal{N} = 9 \) states;

d) the initial wealth \( W_0 \) is equal to 1 at the date 15-Sep-2008.

We perform a comparison to evaluate the impact of the Stable Paretian approximation by comparing the ex-post performance of different portfolio strategies based on: the OA-Sharpe ratio (3), the OA-Asymptotic Sharpe ratio (4), the OA-Stable loss ratio (6), the OA-Stable stochastic bounds ratio (5). Even in this analysis we preselect assets among all those active either in the last ten years or in the last six months. Then we approximate the returns to reduce the randomness of the problem.

For each strategy, we have to compute the optimal portfolio composition 26 times \(^2\) and at the \( k \)-th optimization (\( k = 0, 1, 2, \ldots, 25 \)), two main steps are performed to compute the ex-post final wealth:

**Step 1** Determine the market portfolio \( x_M^{(k)} \) that maximizes the performance ratio \( \rho(W(x)) \) associated to the strategy, i.e. the "ideal" solution of the following optimization problem:

\[
\max_{x^{(k)}} \rho(W(x^{(k)})) \\
\text{s.t.} \quad (x^{(k)})^\prime e = 1, \quad x_i^{(k)} \leq 0.1; \quad x_i^{(k)} \geq 0; \quad i = 1, \ldots, n
\]

Angelelli and Ortobelli (2009) have observed that the complexity of the portfolio problem is much higher in view of a Markovian evolution of the wealth process. In order to overcome this limit we use the Angelelli and Ortobelli’s heuristic algorithm that could be applied to any complex portfolio selection problem that admit more local optima.

**Step 2** During the period \( [t_k, t_{k+1}] \) (where \( t_{k+1} = t_k + T \)) we have to recalibrate daily the portfolio maintaining the percentages invested in each asset equal to those of the market portfolio \( x_M^{(k)} \). Thus, the ex-post final wealth is given by:

\[
W_{t_{k+1}} = W_{t_k} \left( \Pi_{i=1}^{T} \left( x_M^{(k)} \right)^\prime z_{(t_{k+i})}^{(ex post)} \right),
\]

where \( z_{(t_{k+i})}^{(ex post)} \) is the vector of observed daily gross returns between \( (t_k + i - 1) \) and \( (t_k + i) \).

Steps 1 and 2 are repeated for all performance ratios until some observations are available.

The output of this analysis is given by Figures 1 and 2. Figures 1 and 2 report the results of all strategies applied to the preselected assets among all the active in the last 10 years and in the last six months.

### 4 Conclusions

The comparison proposed in this paper confirms that
the recent entries in the market have an important impact in the portfolio choices. As a matter of fact, the results obtained from the stable Paretoian strategies (i.e., OA-Asymptotic Sharpe ratio (4), the OA-Stable loss ratio (6), the OA-Stable stochastic bounds ratio) applied to preselected assets among all the active in the last six months present outstanding results considering that we apply the model during a period of global crisis. However, these results do not consider the transaction costs which must be payed daily in order to maintain constant the percentages invested in each asset. Moreover from this comparison it is still clear that the OA asymptotic Sharpe strategy, as all the other stable Paretoian strategies, presents higher final wealth than the OA Sharpe strategy applied to the preselected assets among all active ones (either in the last ten years or in the last six months).

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