Analysis of Electromagnetic Waves Using the Explicit Group WE-FDTD Method

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Abstract: In this paper, a new explicit group method for scalar wave finite difference time domain (WE-FDTD) is presented to simulate the propagation of two dimensional electromagnetic waves. This method is derived using the implicit Crank-Nicolson finite difference approximation that can be easily converted to explicit form. It is found that the presented method has no longer restricted by stability conditions, improves the accuracy of the conventional finite difference time domain (FDTD) method and provides significant savings in the computational time. Some numerical experiments are included to demonstrate the performance of the presented explicit group method.

Key–Words: FDTD, WE-FDTD, explicit group, Crank-Nicolson, electromagnetic waves.

1 Introduction

Finite Difference Time Domain (FDTD) method is one of the most commonly used numerical methods for the simulation of wave propagation. This method, known as Yee’s algorithm, computes the field components by discretizing the Maxwell’s curl equations both in time and space, and then solving the discretized equations in a time marching sequence by alternatively calculating the electric and magnetic fields in the computational domain ([1],[5]).

Recently, a reduced scalar version of the FDTD method was developed by Aoyagi et. al [2]. In comparison with the FDTD method, the new version called the scalar wave equation finite difference time domain (WE-FDTD) requires less computation and storage. Since then, various techniques have been developed to improve the scalar WE-FDTD computation efficiency ([7]). However, as both the FDTD and WE-FDTD methods are based on an explicit finite difference algorithm, the Courant-Friedrich-Levy (CFL) stability condition must be satisfied. A maximum time step size is limited by the minimum cell size in a computational domain. To overcome this problem, implicit methods must be employed with have no limit on the time-step size arising from the stability consideration. The implicit Crank-Nicolson (CN) method is the most commonly used unconditionally stable method for solving partial differential equations [4]. However, in each time step a global system of equations has to be solved of which will need more simulation time. Improvements have been made to reduce the simulation time when solving large system of equations in partial differential equations. In particular, Evans [3] skillfully developed the explicit group method for solving parabolic equations, Burger equations, diffusion equations, etc. The method can be implemented on parallel computers due to their explicit nature. In this paper, we extend the concept of the explicit group method to simulate the propagation of two dimensional electromagnetic waves that arises in many engineering applications. The explicit group formulation is derived using implicit Crank-Nicolson (CN) finite difference scheme, which is proven to be unconditionally stable. The rest of this papers is organized as follows. In section 2, we develop the formulation of explicit group scheme and its implementation on two dimensional transverse magnetic (TM) wave equation. Section 3 demonstrates the results of some numerical experiments which shows the effectiveness of the presented scheme in terms of stability and accuracy. Finally, conclusions are included in section 4.

2 Formulation

Consider a two-space dimensional electromagnetic problem given by the transverse magnetic (TM) waves
where the field components $E_z$, $H_x$ and $H_y$ exist.

\[
\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_z}{\partial y} \tag{1}
\]

\[
\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_z}{\partial x} \tag{2}
\]

\[
\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \tag{3}
\]

To reduce the algorithm complexity in the TM waves formulations, the equations (1-3) can be solved simultaneously in source free region [2] given by the scalar wave-equation as

\[
\frac{\partial^2 E_z}{\partial t^2} = c_0^2 \left[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} \right] \tag{4}
\]

where $c_0$ is the speed of light in free space medium. Equation (1) forms the basics of the WE-FDTD method which can be discretized in many ways. The standard difference equation known as the implicit Crank-Nicolson for WE-FDTD (1) in uniform grid size $\Delta = \Delta x = \Delta y$ is obtained as

\[
\frac{\delta t}{\Delta t^2} = \frac{c_0^2}{2\Delta} \{ \delta x^2 + \delta y^2 \}
\]

where $u_{i,j}^n = E_z(x,y,t)$ and $\delta$ is the central difference approximation operator such that

\[
\delta t = u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}
\]

\[
\delta x^2 = u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}
\]

\[
\delta y^2 = u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}
\]

After simplification, the equivalent equation is

\[
\alpha u_{i,j}^{n+1} - \beta [u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}]
\]

\[
= 2u_{i,j}^n - \alpha u_{i,j}^{n-1} + \beta [u_{i+1,j}^{n-1} + u_{i-1,j}^{n-1} + u_{i,j+1}^{n-1} + u_{i,j-1}^{n-1}] \tag{5}
\]

where $\alpha = (1 + 2\lambda^2)$, $\beta = \frac{\lambda^2}{2}$ and $\lambda = \left( \frac{c_0 \Delta t}{\Delta x} \right)$ is a CFL factor that determines the stability of the method. It has been established that the scheme above is unconditionally stable with principal error of order $O(\Delta t^2 + \Delta x^2)$. The computational molecule is shown in figure 1 with natural ordering strategy.

### 2.1 Explicit Group WE-FDTD Method

We develop a solving formula for any group of four points $A(i,j), B(i+1,j+1), C(i+1,j), D(i,j+1)$ using finite difference formula (5). This will result in a $4 \times 4$ implicit system matrix form given by

\[
(I + \beta \mathbf{A}) \mathbf{u} = (b_1, b_2, b_3, b_4)^T
\]

![Figure 1: Computational molecule of the Crank-Nicolson scheme with natural ordering.](image)

such that

\[
\mathbf{u} = (u_{i,j}^{n+1}, u_{i+1,j}^{n+1}, u_{i,j+1}^{n+1}, u_{i+1,j+1}^{n+1})
\]

\[
\mathbf{A} = \begin{bmatrix}
4 & -1 & 0 & -1 \\
-1 & 4 & -1 & 0 \\
0 & -1 & 4 & -1 \\
-1 & 0 & -1 & 4
\end{bmatrix}
\]

where

\[
b_1 = 2u_{i,j}^n - \alpha u_{i,j}^{n-1} + \beta [u_{i+1,j}^{n-1} + u_{i-1,j}^{n-1} + u_{i,j+1}^{n-1} + u_{i,j-1}^{n-1}]
\]

\[
b_2 = 2u_{i+1,j}^n - \alpha u_{i+1,j}^{n-1} + \beta [u_{i+2,j}^{n-1} + u_{i,j}^{n-1} + u_{i+1,j+1}^{n-1} + u_{i+1,j-1}^{n-1}]
\]

\[
b_3 = 2u_{i+1,j+1}^n - \alpha u_{i+1,j+1}^{n-1} + \beta [u_{i+2,j+1}^{n-1} + u_{i,j+1}^{n-1} + u_{i+1,j+2}^{n-1} + u_{i+1,j}^{n-1}]
\]

\[
b_4 = 2u_{i,j+1}^n - \alpha u_{i,j+1}^{n-1} + \beta [u_{i-1,j+1}^{n-1} + u_{i,j+2}^{n-1} + u_{i+1,j+1}^{n-1} + u_{i,j+1}^{n-1}]
\]

The system can be written in explicit form as

\[
\mathbf{u} = \frac{1}{\gamma} \begin{bmatrix}
c_1 b_1 + c_2 b_2 + c_3 b_3 + c_4 b_4 \\
c_2 b_1 + c_1 b_2 + c_3 b_3 + c_4 b_4 \\
c_3 b_1 + c_2 b_2 + c_1 b_3 + c_4 b_4 \\
c_4 b_1 + c_3 b_2 + c_2 b_3 + c_1 b_4
\end{bmatrix}
\]
Figure 2: Group Explicit Complete (GEC) strategy with natural ordering at time $n + 1$.

where

\[
\gamma = 1 + 8r + 23r^2 + 28r^3 + 12r^4 \\
c_1 = 1 + 6r^2 + 23r^2 + 7r^3 \\
c_2 = \frac{r^2}{2} + 2r^2 + 2r^3 \\
c_3 = \frac{r^2}{2} + r^3
\]

and $r = \lambda^2$. The explicit group formula (6) is implemented iteratively until a convergence criteria is met. For simplicity, the computational domain is divided into an odd number of squares in both space directions. This gives at every time level \((m-1)^2\) complete groups of four points known as group explicit complete (GEC) strategy as shown in figure 2. Therefore the algorithm of the explicit group WE-FDTD method can be summarized as:

- At each time step
  a) Calculate the solution of each group points using formula (6)
  b) Implement the relaxation procedure.
  c) Check convergence. If converge stop iteration, otherwise re-initialise and do iteration.
  d) Update values

3 Numerical Experiments

In this section, we demonstrate the results of some numerical experiments which show the effectiveness of the explicit group denote as EG WE-FDTD method in terms of stability and accuracy. The experiments were carried out for TM case using a Sun-Fire v240 machine with one processor running in free space medium with normalized electric permittivity and magnetic permeability ($\varepsilon_0 = \mu_0 = 1$).

3.1 Experiment 1

We consider a wave propagation problem defined by (1) where the solution region is set as $\Omega = [0, 1] \times [0, 1]$ surrounded by perfectly electrically conduction (PEC) boundary conditions. The exact solution of the problem is known as:

\[
E_z(x, y, t) = \sqrt{2} \cos(\sqrt{2}\pi t) \sin[\pi(1-x)] \sin[\pi(1-y)]
\]

Tables (1-3) display the simulation results of experiment 1 using different grid sizes and courant factors ($\lambda$) after 30 time steps. We compare the results simulated by EG WE-FDTD method with the available results obtained from the conventional FDTD [6] and standard point Crank-Nicolson (2) methods in terms of errors and CPU time.

From table 1, it is clear that the errors obtained from the EG scheme are smaller than the other methods. This indicates that the EG WE-FDTD method is successful in improving the accuracy of the FDTD and CN WE-FDTD methods. Table 2 shows the superiority of the EG WE-FDTD method, which has the highest accuracy and fastest computational efficiency. From Table 3, the accuracies of both EG WE-FDTD and CN WE-FDTD methods are very close as the value of $\lambda$ increases. However, the EG WE-FDTD method can save more CPU time than the other method, up to 50% or more.

3.2 Experiment 2

A simple Gaussian pulse excitation source is generated at the middle of two dimensional TM waves comprised of a uniform grid cells $60 \times 60$. The computational domain is truncated using perfectly electrically conduction (PEC) boundary conditions. Figures (3-5) demonstrate the simulation results of a Gaussian pulse for the first 50 time steps using courant factors $\lambda = \{0.5, 2.0, 4.0\}$ respectively. From the figures, we can observed that the EG WE-FDTD method gives stable results for $\lambda > 0$.

3.3 Experiment 3

A broadband raised cosine (RC) pulse 2 cycles and a carrier frequency of 10 GHz propagating in vacuum is modelled. The cell size is chosen to provide 20 grid cells per wavelength at center frequency. For simplicity, the normalized time step is taken as 0.5 and the computational domain is truncated using the
Figure 3: $E_z$ field distribution for a Gaussian pulse using EG WE-FDTD method after 50 time steps with $\lambda = 0.5$.

Figure 4: $E_z$ field distribution for a Gaussian pulse using EG WE-FDTD method after 50 time steps with $\lambda = 2.0$.

Figure 5: $E_z$ field distribution for a Gaussian pulse using EG WE-FDTD method after 50 time steps with $\lambda = 4.0$.

Table 1: Comparison results of experiment 1 in terms of average absolute error (A.A.E) and CPU time for $\lambda = 0.5$, after 30 time steps

<table>
<thead>
<tr>
<th>Grid size</th>
<th>Methods</th>
<th>A.A.E</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>FDTD</td>
<td>8.696e-3</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>CN WE-FDTD</td>
<td>1.757e-4</td>
<td>0.031</td>
</tr>
<tr>
<td></td>
<td>EG WE-FDTD</td>
<td>1.662e-4</td>
<td>0.031</td>
</tr>
<tr>
<td>81</td>
<td>FDTD</td>
<td>5.932e-3</td>
<td>0.002</td>
</tr>
<tr>
<td></td>
<td>CN WE-FDTD</td>
<td>8.146e-5</td>
<td>0.047</td>
</tr>
<tr>
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<td>EG WE-FDTD</td>
<td>7.466e-5</td>
<td>0.047</td>
</tr>
<tr>
<td>101</td>
<td>FDTD</td>
<td>1.047e-3</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>CN WE-FDTD</td>
<td>3.936e-5</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>EG WE-FDTD</td>
<td>3.298e-5</td>
<td>0.078</td>
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<tr>
<td>129</td>
<td>FDTD</td>
<td>6.423e-4</td>
<td>0.015</td>
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<tr>
<td></td>
<td>CN WE-FDTD</td>
<td>3.936e-5</td>
<td>0.093</td>
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<tr>
<td></td>
<td>EG WE-FDTD</td>
<td>1.229e-5</td>
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perfectly electrically conduction (PEC) boundary condition. The numerical results obtained by EG WE-FDTD method is presented in figures (6-8) for different time steps. It is found that the numerical solutions computed using the EG WE-FDTD and FDTD methods are in good agreement. This confirm that the EG WE-FDTD method can be used efficiently in solving this kind of scattering problem.
Table 2: Comparison results of experiment 1 in terms of average absolute error (A.A.E) and CPU time for $\lambda = 1.0$, after 30 time steps

<table>
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<th>Grid size</th>
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<td>65</td>
<td>FDTD</td>
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<td>-</td>
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<td></td>
<td>CN WE-FDTD</td>
<td>1.164e-3</td>
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<td>81</td>
<td>FDTD</td>
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<td>-</td>
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<td>1.427e-4</td>
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4 Conclusion

The explicit group scheme derived from the standard Crank-Nicolson approximation for scalar wave finite difference time domain (WE-FDTD) method was presented to simulate the propagation of two dimensional electromagnetic waves. This explicit group known as EG-WE-FDTD method has no longer restricted by stability conditions and provides significant savings in the computational time. The numerical simulations using EG WE-FDTD method are found to be as good as the conventional FDTD method. This suggests that the EG-WE-FDTD method may be a good alternative solver for electromagnetic wave propagations due to its stability and computational efficiency. Since the method is in explicit manner, its efficient implementation on parallel computer is obvious. Report of parallel implementation of EG WE-FDTD method will be reported soon.

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Table 3: Comparison results of experiment 1 in terms of average absolute error (A.A.E) and CPU time for $\lambda = 4.0$, after 30 time steps

<table>
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<th>Methods</th>
<th>A.A.E</th>
<th>CPU time (s)</th>
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<td>0.485</td>
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<td>EG WE-FDTD</td>
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<td>1.172</td>
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Figure 7: $E_z$ field for a broadband RC pulse using EG WE-FDTD method after 120 time steps.

References:


Figure 8: $E_z$ field for a broadband RC pulse using EG WE-FDTD method after 240 time steps.
