Masonry domes: Equilibrium states and structural assessment

ALESSANDRO BARATTA
Department of Civil, Environmental, Architectural Engineering,
University of Naples "Federico II"
Via Claudio 21, 80125 - Napoli
ITALY
alessandro.baratta@unina.it

Abstract: - Basic properties of masonry do not allow to rely on tensile strength, and flexural strength cannot be trusted on. Nevertheless in 2D walls and in double curvature vaults, a particular organization of the vault apparatus can in some instances, through the action of compression and friction, give place to a equilibrium pattern including tension, which explains the unexpected good performance of some walls and cupolas

Key-Words: - Masonry behaviour, Double curvature vaults, Cupolas, No-Tension material, Masonry texture, Structural assessment

1 Introduction

Masonry is the main material mankind has exploited to provide itself a shelter.

Homes, temples, offices, markets and so on are built by some kind of masonry since the beginning of civilization. Walls are the main way loads are transferred to foundations and to underlying soil, but horizontal floor structures require some more skill, since masonry, due to its very poor, unreliable, inhomogeneous and time-degrading, tensile strength, is not able to resist bending moments. This is the reason why masonry buildings are often complemented by wood or, more recently, steel systems to cover spaces, providing beam elements resisting by pure flexure.

Man learned from nature that it was possible to cover spaces with stones. He observed natural arches and found inhabited large caves, so the attempt to reproduce nature (a strong impulse in Architecture, as testified also in recent times by the Gaudi’s opera, see e.g. [1]) may be has pushed to realize double-curvature roofs. This activity gradually resulted in a success, with larger and larger spans being covered, thus leading to the early architecture and to its developments up to our times.

The historical development of Structural Mechanics is exhaustively reconstructed in the book by E. Benvenuto [2]. A very interesting and complete historical survey on the conception, realization and progress in the masonry vaults technology can be found in [3] and in [4,5].

The prevalent feature that characterizes masonry structures, and makes them dissimilar from modern concrete and steel structures, is quite definitely their intrinsic inability to resist tensile stresses. So, it is natural that the material model, that is intended to be an "analogue" of real masonry, cannot resist tensile stress, but, possibly, behaves elastically under pure compression.

No-Tension (NT) solutions for masonry structures (see e.g. [6, 7, 8]) are however a very significant reference point and a powerful tool for reliable structural assessment, for many reasons. The first reason is that the NT model is a stable behaviour, poorly subject to uncertainty and aging. Tensile strength is in any case small, uncertain, highly variable in the mass of a structure, not durable in time and so on. Equilibrium of NT domes and vaults can be approached by the Monge-Ampère equation [9, 10]. On this basis, when a weak or vulnerable structure is encountered, the first action for reinforcement is to provide some tensile strength were necessary. An opportunity is today offered by the application of FRP strips (see e.g. [11, 12]) Anyway neglecting tensile strength leads to a safe assessment. In some cases a surprisingly good performance of masonry buildings is encountered; that can be explained by a particular skill in the apparatus of masonry. In the following, an example of the effectiveness of masonry texture and of its influence on the load carrying capacity of domes is illustrated.

2 Effect of masonry texture

The influence of the texture on the masonry performance can be illustrated by the following example. Assume that a panel is built by regular bricks with interposed poor mortar joints, lacking any adhesive force.
Consider that bricks are set according to the following two patterns (Fig. 1a,b). If there is no vertical compression both panels are free to expand laterally without encountering any resistance (Fig. 1c). If a vertical compression is applied, the panel in Fig. 1a still can freely separate; by contrast an horizontal tensile pseudo-strength becomes active in Fig. 1b, because of friction and interlocking of bricks with each other.

![Masonry element: Aligned bricks; b) Staggered bricks; c) Free lateral expansion for both panels](image)

The failure mechanism in Fig. 2 can be studied for the bi-dimensional masonry plane element in Fig. 1b having a friction coefficient $f$, a joints stagger $s$ (Fig. 2) and a row density $\omega$ defined as the ratio between the number of block rows in the panel height $H$ and the height $H$. In Fig. 2, $\omega = 7/H$.

![Masonry element: Failure mechanism under compression and limit tensile forces; stagger parameter](image)

The wall is subjected to vertical compression stresses $\sigma_y$ orthogonal to the joints direction and horizontal tractions $\sigma_x$ parallel to the joints. It is possible to prove [13] that the horizontal tensile strength $\sigma'_\alpha$ is given by (Fig. 2)

$$\sigma'_\alpha = -f\sigma_y \omega$$  (1)

The ratio between the compressive stress on the joints and the transverse tensile strength is

$$\frac{\sigma'_\alpha}{\sigma_y} = f\omega$$  (2)

If the length of the stone is $a$, $s$ is of the order $a/2$. Usually $a > 2h$ (very often $a > 4h$), with $h$ the thickness of the brick, and so $s > h$. On the other side, $\omega \approx 1/h$, so that $s\omega > 1$ (very often $s\omega > 2$). With the help of mortar and/or of roughness of the interface between stones, $f$ may possibly be rather large ($f = 0.5 \div 0.8$), and the ratio in Eq. (2) is frequently larger than 1, i.e. the tensile strength in the direction parallel to joints is larger than the acting compressive stress.

It can be also proved that a pretty ductility is associated to the tensile strength $\sigma'_\alpha$. With reference to the diagrams in Fig. 3, applying a safety coefficient $\gamma$ to the limit resistance $\sigma'_\alpha$, the loss in strength is balanced by a gain in ductility. In other words if $\sigma'_a$ is the admissible stress and $\delta_a$ is the maximum ductility, one can write

$$\sigma'_a = \sigma'_\alpha / \gamma$$  
$$\delta_a = \frac{\varepsilon'_a}{\varepsilon'_{\alpha}} = 1 + \frac{\varepsilon'_a}{\varepsilon'_\alpha}(\gamma - 1)$$  (3)

![Stress vs. deformation in the tension range, conventional diagram with variable ductility](image)

A fundamental observation is that Eq. (1) not only expresses the tensile resistance of the masonry element, but also puts to evidence that the tension can be contrasted in function of the static needs by means of a skilled orientation of the texture of the masonry blocks and of the mortar joints. After recognizing that by the combined effect of compression and friction the lines of the mortar joints are probably the lines where original designers and builders intended to provide tensile strength in the masonry mass, it can be conceived that a technical practice had spread out, very similar to the modern technology of reinforced concrete where the structural designer inserts steel bars in
way to balance tension along stretched lines.
Many examples proving that clever architects were aware of this effect when designing vault structures can be illustrated as for instance in so-called cantilever stairs (see e.g. [14]).

3 Tension in spherical domes
Consider the axial-symmetric hemispherical dome with radius $R$ and thickness $t$ (Fig. 4a), supporting its own weight $w$, where it is well known that in the classical solution, tension should be active along the parallel lines after some degree of the zenith angle $\phi = 51.8^\circ$.

![Fig. 4: a) Spherical dome; b) Ratio of parallel to meridian stress](image)

The ratio is

$$\frac{N_0}{N_\phi} = 1 - \cos \phi - \cos^2 \phi \quad (5)$$

The ratio is plotted in Fig. 4b, whence one can see that the ratio is always not larger than 1. So, if masonry is organized by staggered regular bricks – as often happens – tension could generally be faced by the friction mechanism as illustrated in Sec. 2 (Fig. 4c).

Anyway, equilibrium can be found by some other membrane surface other than the mean surface of the shell, provided it is included in the thickness between the (spherical) intrados and extrados.

Considering a revolution membrane surface having an elliptic profile with radii $a$ and $b$, included in the interior of the hemisphere (Fig. 5a) the internal forces equilibrating the weight of the spherical dome can be found as follows

![Fig. 5: a) The elliptic membrane surface included in the dome thickness; b) Possible physiological fractures](image)

The radii of curvature of the ellipsoidal surface

Consider the spherical cap above the center angle $\beta$, whose weight is

$$W = 2\pi w R^2 (1 - \cos \beta) \quad (6)$$

The angle $\beta$ is related to the zenith angle $\phi$ by the relationships

$$\sin \beta = \frac{a \sin \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}$$
$$\cos \beta = \frac{b \cos \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}$$
$$d\beta = \frac{ab}{a^2 \sin^2 \phi + b^2 \cos^2 \phi} \cos \phi \sin \beta \cos \phi \, d\phi$$

The radii of curvature of the ellipsoidal surface
are ([15], p. 40)

\[ r_1(\varphi) = \frac{a^2 b^2}{\left(\frac{a^2 \sin^2 \varphi + b^2 \sin^2 \varphi}{2}\right)^{1/2}} \]

\[ r_2(\varphi) = \frac{a^2}{\left(\frac{a^2 \sin^2 \varphi + b^2 \sin^2 \varphi}{2}\right)^{1/2}} \]

so that

\[ \sin \beta = \frac{a r_2(\varphi) \sin \varphi}{a^2} = \frac{r(\varphi)}{a} \]

\[ \cos \beta = \frac{b \ r(\varphi)}{a \ a \tan \varphi} \quad (9) \]

\[ r(\varphi) = r_2(\varphi) \sin \varphi \]

and

\[ d\beta = \frac{a \ r_1(\varphi)}{b \ r_2(\varphi)} d\varphi \quad (10) \]

The equilibrium versus the vertical translation can be written

\[ 2\pi r N_\varphi(\varphi) \sin^2 \varphi + W = 0 \quad (11) \]

and

\[ N_\varphi(\varphi) = -\frac{wR^2(1-\cos \beta)}{r_2(\varphi) \sin^2 \varphi} \quad (12) \]

The ellipsoidal membrane shall now sustain the weight \( w \) of the spherical shell, that transforms in the weight \( w^* \) on the ellipsoid setting

\[ w^* r_1(\varphi) d\varphi = wR \beta r_2 d\theta \quad r_2 = R \sin \beta \quad (13) \]

whence

\[ w^* = wR^2 \frac{1}{br_2(\varphi)} \quad (14) \]

The equilibrium along the outward normal to the (ellipsoidal) membrane yields

\[ \frac{N_\varphi(\varphi)}{r_1(\varphi)} + \frac{N_\theta(\varphi)}{r_2(\varphi)} = p_n(\varphi) = -w^* \cos \varphi = -w \frac{R^2}{br_2(\varphi)} \cos \varphi \quad (15) \]

and

\[ N_\theta(\varphi) = -\left( w \frac{R^2}{b} \cos \varphi + N_\varphi(\varphi) \frac{r_2(\varphi)}{r_1(\varphi)} \right) \quad (16) \]

Ellipsoidal stress surface can be active in order to mitigate tension hoop stresses, possibly after some fractures have opened (Fig. 5b), that can be considered physiological if masonry has some degree of ductility in the parallel direction, as in the friction strength mechanism illustrated in Sec. 2. In Fig. 6a various membrane stress surfaces are plotted, with different ratios \( a/b \).

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Fig. 6: a) Ellipsoidal membrane surfaces for different ratios of the ellipse radii \( a \) and \( b \) to the radius \( R \) of the spherical dome; b) Ratio of \( N_\theta \) to \( N_\varphi \) for different shapes of the elliptic profile.

Note that such surfaces make sense provided that they remain included in the thickness of the spherical shell, i.e. if \( t \geq 2(R-a) \) and \( t \geq 2(b-R) \), with \( b \geq a \), since it is assumed that the interface in the meridian direction is no-tension. The plots in Fig. 6b prove that the ratio of the parallel to the meridian normal force can be mitigated, and also be near 0.4 and smaller, with increasing the ratio \( b/a \), a value that is very often in the range of the ratio \( \sigma_{ts}/\sigma_t \) in Eq. (2), so that one can conclude that tensile hoop stress most times does not cause any problem. Consider that both in the spherical and in the elliptic membranes, the stress surface is a complete semi-ellipsoid, with \( \varphi = 90^\circ \) at \( \gamma = 0 \), so that the equilibrium solutions do not require any thrust force at the bottom support \( \gamma = 0 \).

Anyway, it has been proved by [16] that a membrane surface included in the thickness of the dome can be found without hoop tension, provided that a adequate counter-thrust force can be exerted at the bottom of the dome. In Fig. 7a it is illustrated how the spherical and elliptic membranes only transfer vertical actions on the basement, \( v_s \) and \( v_e \).
respectively, while a no-tension profile requires that the base support can support a horizontal force \( h_n \) (Fig. 7b).

Fig. 7: No-thrust and no-tension stress surfaces: a) The basement of the dome is not subject to thrust action, but lower parallel lines are under tension; b) If a no-tension solution is adopted, the support of the dome is subject to a horizontal thrust force. Tension in the parallel lines is transferred to the basement.

4 The “masonry apparatus”. An help to intuition.

Reading masonry texture in a vault can help in understanding its equilibrium asset. The first element is indeed its geometry, a cross vault yields an equilibrium pattern different than a barrel vault, and so on. But a double-curvature surface, apart from its particular conception is anyway a highly hyperstatic system, and the equilibrium is never uniquely determinate. So the way the stones are jointed all together is a key to understand what equilibrium path would stresses run through, and/or what path would the builder have preferred to drive the vault into accommodate in.

So, consider for instance the two vaults in Fig. 8a and in Fig. 8b, having the same geometry, but in vault a) the mortar rows are parallel to the base perimeter, while in the vault b) the mortar rows are normal to the perimeter. The postulate is that compression normal to the mortar rows is the preferred equilibrium path for the vault, and that this is the tool for the original builder to steer the vault into a (his own) objective static asset. If the preferred direction for compression is normal to the perimeter, it is expected that compression acts along the arrows drawn in Figs. 8, a) and b), so that the vault gains a tensile capacity in the direction orthogonal to the arrows. It is easy to understand that this produces an effect on the thrust the vault exerts on the base supports. Consider in fact that in both cases the vault is made by four gores. In the case a) compression is directly transferred to the sides of the basement, while lateral dilatation and the diffusion of stresses to the corners is contrasted by internal tensile strength; so two opposite gores tend to directly sustain each other, and the distribution of the horizontal thrust force tends to concentrate towards the middle of the sides (Fig. 8c). By contrast, in case b) compression is active in the direction parallel to the base sides, and the gores tend to support each other along the diagonal lines, while the orthogonal dilatation and diffusion of stress are now contrasted by tensile strength in the direction orthogonal to the sides; so all forces tend to converge in the corners, and the distribution of the horizontal thrust force tends to concentrate to the corners (Fig. 8d).

Fig. 8: Influence of the vault apparatus on the static behaviour of vaults. The difference in the apparatus in Figs. a) and b) yields different equilibrium pattern and a different distribution of the thrust force as in Figs. c)- d).

In other words, by acting on the masonry apparatus it is possible that, with the same geometry, a structure may be realized that works like a cloister vault rather than like a groin vault or viceversa. Which means that it may be not wise to analyze the statics of a vault only on the basis of its geometry. Anyway, a skilled design of apparatus is also a tool to build vaults without formworks [17].

5 Conclusions

Historical masonry vaults and/or cupolas exhibit a large variety of typological assets. Often masonry is well operated, with strong stones and effectively adhesive mortar; in many cases masonry is in worse working order; in other cases a poor masonry is encountered.

Anyway, double-curvature structures can appeal to many equilibrium patterns to sustain at
least their own weight plus some light additional loads. So they are, in general, stable systems, provided that their supports are strong and able to contrast thrust forces. Vaults are in general characterized by their shape, and a lot of types can be listed (see e.g. [18]), that have been conceived to be included in any simple or complex architectural design. But the equilibrium paths are also driven by the way masonry is interwoven. In some cases, a masterly design of the masonry tissue and of the vault apparatus may help in improving the structure's stability, and sometimes even in preventing fractures, as discussed and illustrated in Sec. 3.

It should be realized, by contrast, that fractures are almost always a physiological feature of masonry; since almost always it has not significant tensile strength, it cannot expand by tension and, when necessary to comply with congruence of the overall deformation, dilatation is provided by fractures.

References:


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