Nonlinear Optimal Tracking Using Finite-Horizon State Dependent Riccati Equation (SDRE)

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Abstract: A number of computational techniques have been proposed for synthesizing nonlinear control based on state dependent Riccati equation (SDRE). Most of these techniques focusing on infinite-horizon regulator problems. This paper presents a new and computationally efficient approximate online technique used for finite-horizon nonlinear tracking problems. This technique based on change of variables that converts the differential Riccati equation to a linear Lyapunov differential equation. Illustrative examples are given to show the effectiveness of the proposed technique.

Key–Words: Optimal control, finite-horizon state dependent Riccati equation, nonlinear tracking

1 Introduction
The need to improve performance in controlled systems leads to more and more accurate modeling. However, if a model is a good representation of the real system over a wide range operating points, it is most often nonlinear. Therefore, the ordinarily used linear control techniques become inadequate and it becomes necessary to use some other nonlinear control techniques. And the competitive era of rapid technological change has motivated the rapid development of nonlinear control theory for application to challenging complex dynamical real-world problems [1]. There exist many nonlinear control design techniques, each has benefits and weaknesses. Most of them are limited in their range of applicability, and use of certain nonlinear control technique for a specific system usually demands choosing between different factors, e.g. performance, robustness, optimality, and cost. Some of the well-known nonlinear control techniques are feedback linearization, adaptive control, nonlinear predictive control, sliding mode control, and approximating sequence of Riccati Equations. One of the highly promising and rapidly emerging techniques for nonlinear optimal controllers designing is the SDRE technique.

Inspired by the great potential of SDRE for regulation of infinite horizon nonlinear systems [2], SDRE technique for infinite horizon tracking of nonlinear systems [3], and the development of finite horizon SDRE for finite horizon regulation of nonlinear systems [4], in this paper, we develop the SDRE technique for finite horizon optimal tracking of nonlinear systems based on a change of variable [5], that converts the differential Riccati equation (DRE) to a linear differential Lyapunov equation [6], which are solved in real time at each time step.

The reminder of this paper is organized as follows: An overview of infinite-horizon SDRE technique is discussed in Section II. Section III presents the SDRE in finite horizon regulator problem. Section IV presents the nonlinear finite horizon tracking technique via SDRE. Illustrative examples are discussed in Section V. Finally conclusions of this paper are given in Section VI.

2 Overview of Infinite-Horizon SDRE Technique
SDRE, which is also referred to as the Frozen Riccati Equation (FRE) [7], first proposed by Pearson (1962) and later expanded by Wernli & Cook (1975), and studied by Mracek & Cloutier (1998) [8]. SDRE has become as a very attractive tool for the systematic design of nonlinear controllers, very common within the control community over the last decade, providing an extremely effective algorithm for nonlinear feedback control design by allowing nonlinearities in the system states while additionally offering great design flexibility through design matrices [1]. The method involves factorization of the nonlinear dynamics into product of a matrix-valued function. Thus, the SDRE algorithm captures the nonlinearities of the system,
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advantage offered by SDRE to the control designer is
be carried out online, in which case the SDRE is de-
depends only on the current state, the computation can
stabilizing solution during state evolution yields the
a given point in state space, a SDRE whose point wise
made to be very close to optimal solution. The coeffi-
tients of this Riccati equation vary with the each point
in state space. The algorithm thus involves solving, at
a given point in state space, a SDRE whose point wise
stabilizing solution during state evolution yields the
SDRE nonlinear feedback control law. As the SDRE
depends only on the current state, the computation can
be carried out online, in which case the SDRE is de-
defined along the state trajectory. In addition, a primary
advantage offered by SDRE to the control designer is
the opportunity to make tradeoffs between control ef-
fort and state errors by tuning the SDC.

Fig.1 shows a process of the infinite-horizon SDRE technique. At each sample time, the follow-
ing procedure is accomplished. First, the current state
vector $x$ is used to calculate numerical values for
$A(x)$ and $B(x)$. Then, using the LQR equations, $P$
and $K$ are calculated. Control input $u$ is then calcu-
lated and applied to the system. This procedure is then
repeated at the next sample time. For the SDRE tech-
nique, the ARE is solved at every sample time for each
new value of $A(x)$ and $B(x)$. This causes the nonlin-
er system to be approximated as a series of linear
systems. Therefore, shorter time increments increase
the accuracy of the control law, because this decreases
the amount of time that each approximation is applied.
Because of its approximating nature, the SDRE tech-
nique is considered a suboptimal solution. However,
with the proper choices for the $A(x)$ and $B(x)$ matri-
ces, and with the proper amount of sample times, the
SDRE technique can provide a very adequate optimal
solution.

3 Nonlinear Finite-Horizon Regula-
tor Via SDRE

3.1 Problem Formulation
The nonlinear system considered in this paper is as-
sumed to be in the form:

$$
\dot{x}(t) = f(x) + g(x)u(t), \quad (1)
$$
$$
y(t) = h(x). \quad (2)
$$

That nonlinear system can be expressed in a state-
dependent like linear form, as:

$$
\dot{x}(t) = A(x)x(t) + B(x)u(t), \quad (3)
$$
$$
y(t) = C(x)x(t), \quad (4)
$$

where $f(x) = A(x)x(t)$, $B(x) = g(x)$, $h(x) = C(x)x(t)$.

The goal is to find a state feedback control law of
the form $u(x) = -kx(t)$, that minimizes a cost function
given by [10]:

$$
J(x, u) = \frac{1}{2}x'(t_f)Fx(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x'(t)Q(x)x(t) + u'(x)R(x)u(x)] dt,
$$

where $Q(x)$ and $F$ is a symmetric positive semi-
definite matrix, and $R(x)$ is a symmetric positive def-
inite matrix. Moreover, $x'Q(x)x$ is a measure of
control accuracy and $u'(x)R(x)u(x)$ is a measure of
control effort.

3.2 Approximate Solution for Finite-
Horizon SDRE Regulator
To minimize the above cost function (5), a state feed-
back control law can be given as

$$
u(x) = -kx(t) = -R^{-1}(x)B'(x)P(x)x(t), \quad (6)
$$

where $P(x, t)$ is a symmetric, positive-definite solu-
tion of the state-dependent Differential Riccati Equa-
tion (SDDRE) of the form.
\[ -\dot{P}(x) = P(x)A(x) + A'(x)P(x) - P(x)B(x)R^{-1}(x)B'(x)P(x) + Q(x), \]

with the final condition
\[ F = P(x, t_f). \]

The resulting SDRE-controlled trajectory becomes the solution of the state-dependent closed-loop dynamics
\[ \dot{x}(t) = [A(x) - B(x)R^{-1}(x)B'(x)P(x)]x(t) \]

As the SDRE function of \((x, t)\), we do not know the value of the states ahead of present time step. Consequently, the state dependent coefficients cannot be calculated to solve (7) with the final condition (8) by backward integration from \(t_f\) to \(t_0\). To overcome the problem, an approximate analytical approach is used [4, 5, 6], which converts the original nonlinear Riccati equation to a differential Lyapunov equation. At each time step, the Lyapunov equation can be solved in closed form. In order to solve the DRE (7), one can follow the following steps at each time step:

- Solve Algebraic Riccati Equation (ARE) to calculate the steady state value \(P_{SS}(x)\)
  \[ P_{SS}(x)A(x) + A'(x)P_{SS}(x) - P_{SS}(x)B(x)R^{-1}(x)B'(x)P_{SS}(x) + Q(x) = 0. \]

- Use changing of variables technique and assume that \(K(x, t) = [P(x, t) - P_{SS}(x)]^{-1}. \)
- Calculate the value of \(A_{cl}(x)\) as
  \[ A_{cl}(x) = A(x) - B(x)R^{-1}(x)B'(x)P_{SS}(x). \]
- Solve the algebraic Lyapunov equation [11]
  \[ A_{cl}D + DA_{cl}' - BR^{-1}B' = 0. \]

- Solve the differential Lyapunov equation
  \[ \dot{K}(x, t) = K(x, t)A_{cl}(x) + A_{cl}(x)K(x, t) - B(x)R^{-1}(x)B'(x). \]

The solution of (12), as shown by [12], is given by
\[ K(t) = e^{A_{sl}(t-t_f)}(K(x, t_f) - D)e^{A_{sl}'(t-t_f)} + D. \]

- Calculate the value of \(P(x, t)\) from the equation
  \[ P(x, t) = K^{-1}(x, t) + P_{SS}(t). \]

- Finally, calculating the value of the optimal control \(u(x, t)\) as
  \[ u(x, t) = -R^{-1}B'(x)P(x, t)x. \]

## 4 Nonlinear Finite-Horizon Tracking Using SDRE

### 4.1 Problem Formulation

Consider nonlinear system given in (1) and (2), which can be redescibed in the form (3) and (4). Let \(z(t)\) be the desired output.

The goal is to find a state feedback control law that minimizes a cost function given by:

\[ J(x, u) = \frac{1}{2}e'(t_f)Fe(t_f) \]

\[ + \frac{1}{2} \int_{t_0}^{t_f} [e'(t)Qe(t) + u'(t)R(u)u(t)] dt, \]

where \(e(t) = z(t) - y(t). \)

### 4.2 Approximate Solution for Finite-Horizon SDDRE Tracking

To minimize the above cost function (16), a feedback control law can be given as

\[ u(x) = -R^{-1}B'(x)[P(x)x - g(x)], \]

where \(P(x)\) is a symmetric, positive-definite solution of the state-dependent Differential Riccati Equation (SDDRE) of the form

\[ -\dot{P}(x) = P(x)A(x) + A'(x)P(x) - P(x)B(x)R^{-1}(x)B'(x)P(x) + C'(x)Q(x)C(x), \]

with the final condition
\[ P(x, t_f) = C'(t_f)FC(t_f). \]

The resulting SDRE-controlled trajectory becomes the solution of the state-dependent closed-loop dynamics

\[ \dot{x}(t) = [A(x) - B(x)R^{-1}(x)B'(x)P(x)]x(t) \]

\[ + B(x)R^{-1}(x)B'(x)g(x), \]

where \(g(x)\) is a solution of the state-dependent non-homogeneous vector differential equation

\[ \dot{g}(x) = -[A(x) - B(x)R^{-1}(x)B'(x)P(x)]'g(x) + C'(x)Q(x)z(x), \]

with the final condition
\[ g(x, t_f) = C'(t_f)Fz(t_f). \]

Similar to Section 3, an approximate analytical approach is used and the DRE can be solved in the following steps at each time step:
5 Illustrative Examples

In this section, two illustrative examples, which have analytical solution for finite-horizon SDDRE tracking control design, are given. The first example is linear tracking and the second example is nonlinear tracking.

5.1 Example 1

Consider the second order system

\[
\dot{x}_1(t) = x_2(t) \tag{27}
\]

\[
\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + u(t), \tag{28}
\]

\[
y(t) = x_1(t), \tag{29}
\]

with the initial conditions

\[
x_0 = [0, 0]' \tag{30}
\]

The reference output is

\[
z(t) = 3, \tag{31}
\]

and the selected weighted matrices are

\[
Q = diag(100, 10), R = 0.4, F = diag(20, 20). \tag{32}
\]

The simulations were performed for final time of 10 seconds and the resulting output trajectory is shown in Fig. 3, and the optimal control is shown in Fig. 4.

In Fig. 3, the solid line denotes the actual output trajectory of the finite-horizon tracking controller, the doted line denotes the reference output signal. It’s clear that the algorithm gives very good result as the actual output is making almost ideal tracking to the reference output and it shows that the algorithm is able to solve the state dependent differential Riccati equation (SDDRE) finite-horizon tracking problem with minimum error.

5.2 Example 2

The dynamic equation for forced damped pendulum is:

\[
ml^2\ddot{\theta} = -mglsin(\theta) - k\dot{\theta} + T, \tag{33}
\]

where, \( \theta \) .. angle of pendulum, \( l \) .. length of rod, \( m \) .. mass of pendulum, \( g \) .. gravitational constant, \( k \) .. damping (friction) constant, \( T \) .. driving torque.

The system nonlinear state equations can be written in the form:

\[
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-a_1 - b_1 \dot{x}_1 - b_2 \dot{x}_2 - b_3 \dot{x}_3 - b_4 \dot{x}_4 \\
b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 \\
-c_1 - d_1 \dot{x}_1 - d_2 \dot{x}_2 - d_3 \dot{x}_3 - d_4 \dot{x}_4 \\
c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} u(t),
\]

where, \( a_i, \dot{a}_i, b_i, \dot{b}_i, c_i, \dot{c}_i, d_i, \dot{d}_i \).. system parameters.

The system state equations can be written in the form:

\[
\dot{\mathbf{x}}(t) = 
\begin{bmatrix}
-a_1 - b_1 \dot{x}_1 - b_2 \dot{x}_2 - b_3 \dot{x}_3 - b_4 \dot{x}_4 \\
b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 \\
-c_1 - d_1 \dot{x}_1 - d_2 \dot{x}_2 - d_3 \dot{x}_3 - d_4 \dot{x}_4 \\
c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} u(t),
\]

\[
\mathbf{x}(t) = 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}.
\]

\[
Q = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}, R = 0.4, F = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}
\]

The simulations were performed for final time of 10 seconds and the resulting output trajectory is shown in Fig. 3, and the optimal control is shown in Fig. 4.

In Fig. 3, the solid line denotes the actual output trajectory of the finite-horizon tracking controller, the doted line denotes the reference output signal. It’s clear that the algorithm gives very good result as the actual output is making almost ideal tracking to the reference output and it shows that the algorithm is able to solve the state dependent differential Riccati equation (SDDRE) finite-horizon tracking problem with minimum error.
\[ \dot{x}_1 = x_2, \quad (33) \]

\[ \dot{x}_2 = -\frac{g}{l} \sin (x_1) - \frac{k}{ml^2} x_2 + \frac{1}{ml^2} u, \quad (34) \]

where: \( \theta = x_1, \dot{\theta} = x_2, T = u. \)

Or alternatively in state dependent form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-4.9 \sin (x_1) & -0.25 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} +
\begin{bmatrix}
0 \\
0.25 \\
\end{bmatrix} u. \quad (35)
\]

Let the reference output is

\[ z(t) = \sin (t), \quad (36) \]

and the selected weighted matrices are

\[ Q = \text{diag}(1000, 0), R = 0.07, F = \text{diag}(1, 1). \quad (37) \]

The simulations were performed for final time of 12 seconds and the resulting output trajectory is shown in Fig. 5, and the optimal control is shown in Fig. 6. In Fig. 5, the solid line denotes the actual output trajectory of the finite-horizon tracking controller, the dotted line denotes the reference output signal.

Comparing these trajectories in Fig. 5, it’s clear that the finite-horizon SDDRE nonlinear tracking algorithm gives very good results as the actual optimal output is making very good tracking to the reference output, and the developed algorithm is able to solve the SDDRE finite-horizon nonlinear tracking problem.

6 Conclusion

The paper offered a new approximate online technique used for finite-horizon nonlinear tracking problems. This technique based on change of variables that converts the differential Riccati equation to a linear Lyapunov equation. During online implementation, the Lyapunov equation is solved in a closed form at the given time step. Two Illustrative examples are in-
cluded to demonstrate the effectiveness of the developed technique.

References:


